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Relations of 0 and ∞

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Abstract. In this paper, as the representation of the point at infinity of the Riemann sphere by the zero $z = 0$, we will show some delicate relations between 0 and ∞ which show a strong discontinuity at the point of infinity on the Riemann sphere. We did not consider any value of the elementary function $W = 1/z$ at the origin $z = 0$, because we did not consider the division by zero $1/0$ in a good way. Many and many people consider its value by the limiting like $+\infty$, $-\infty$ or the point at infinity as ∞. However, their basic idea comes from **continuity** with the common sense or based on the basic idea of Aristotle. However, as the division by zero we will consider its value of the function $W = 1/z$ as zero at $z = 0$. We will see that this new definition is valid widely in mathematics and mathematical sciences. In this paper, we will show its geometrical properties.

1. Introduction

By a natural extension of the fractions

$$\frac{b}{a} \tag{1}$$

for any complex numbers a and b, we found the simple and beautiful result, for any complex number b

$$\frac{b}{0} = 0, \tag{2}$$

incidentally in [12] by the Tikhonov regularization for the Hadamard product inversions for matrices and we discussed their properties and gave several physical interpretations on the general fractions in [5] for the case of real numbers. Meanwhile, Eq. (2) is a very special case of very general fractional functions in [4].

We thus should consider, for any complex number b, as Eq. (2); that is, for the mapping

$$W = \frac{1}{z}, \tag{3}$$

the image of $z = 0$ is $W = 0$ (**should be defined**). This fact seems to be a curious one in connection with our well-established popular image for the point at infinity on the Riemann sphere. Therefore, the division by zero will give great impacts to complex analysis and to our ideas for the space and universe.

However, the division by zero Eq. (2) is now clear, indeed, for the introduction of Eq. (2), we have several independent approaches as in:

1) by the generalization of the fractions by the Tikhonov regularization or by the Moore-Penrose generalized inverse,

2) by the intuitive meaning of the fractions (division) by H. Michiwaki,

3) by the unique extension of the fractions by S. Takahasi, as in [14],

4) by the extension of the fundamental function $W = 1/z$ from $\mathbf{C} \setminus \{0\}$ into $\mathbf{C}$ such that $W = 1/z$ is a one to one and onto mapping from $\mathbf{C} \setminus \{0\}$ onto $\mathbf{C} \setminus \{0\}$ and the division by zero $1/0 = 0$ is a one to one and onto mapping extension of the function $W = 1/z$ from $\mathbf{C}$ onto $\mathbf{C}$, and

5) by considering the values of functions with the mean values of functions.

Furthermore, in [6] we gave the results in order to show the reality of the division by zero in our world:

A) a field structure containing the division by zero - the Yamada field $\mathbf{Y}$,

B) by the gradient of the y axis on the (x, y) plane - $\tan \frac{\pi}{2} = 0$,

C) by the reflection $W = 1/\overline{z}$ of $W = z$ with respect to the unit circle with center at the origin on the complex z plane --- the reflection point of zero is zero,
and

D) by considering rotation of a right circular cone having some very interesting phenomenon from some practical and physical problem.

Furthermore, in [8] and [12], we discussed many division by zero properties in the Euclidean plane. In [7], we gave beautiful geometrical interpretations of determinants from the viewpoint of the division by zero.

See also J. A. Bergstra, Y. Hirshfeld and J. V. Tucker [2] and J. A. Bergstra [3] for the relationship between fields and the division by zero, and the importance of the division by zero for computer science. It seems that the relationship of the division by zero and field structures are abstract in their papers.

Meanwhile, J. P. Barukčić and I. Barukčić ([1]) discussed recently the relation between the divisions $0/0, 1/0$ and special relative theory of Einstein. However, it seems that their results are curious with their logics. Their results contradict with ours.

Furthermore, T. S. Reis and J. A. D. W. Anderson ([9, 10]) extend the system of the real numbers by defining division by zero with three infinities $+\infty, -\infty, \Phi$. Could we accept their theory as a natural one? They introduce a curious ideal number for the division $0/0 = \Phi$.

Meanwhile, we should refer to up-to-date information: *Riemann Hypothesis Addendum - Breakthrough Kurt Arbenz* : https://www.researchgate.net/publication/272022137 .

Here, we recall Albert Einstein's words on mathematics: Blackholes are where God divided by zero. I don’t believe in mathematics. George Gamow (1904-1968) Russian-born American nuclear physicist and cosmologist remarked that "it is well known to students of high school algebra" that division by zero is not valid; and Einstein admitted it as **the biggest blunder of his life** (Gamow, G., My World Line (Viking, New York). p 44, 1970).

See [12], for example, for the mysterious story of the division by zero.

However, for functions, we will need some modification **by the idea of the division by zero calculus**:

For any formal Laurent expansion around $z = a$,

$$f(z) = \sum_{n=-\infty}^{\infty} C_n (z-a)^n, \tag{4}$$

we obtain the identity, by the division by zero

$$f(a) = C_0. \tag{5}$$

The typical example is that, as we can derive by the elementary way, $\tan\frac{\pi}{2} = 0.$

We gave many examples with geometric meanings in [8]. See [4, 5, 6, 12, 13, 14] for the related references. In [8], many division by zero results in Euclidean spaces are given and the basic idea at the point at infinity should be changed. In [7], we gave beautiful geometrical interpretations of determinants from the viewpoint of the division by zero. The results show that the division by zero is our basic and elementary mathematics in our world.

In this paper, we will show the fundamental geometric interrelations between 0 and ∞.

2. $n = 2,1,0$ regular polygons inscribed in a disc

We consider n regular polygons inscribed in a fixed disc with radius a. Then we note that their area S_n and the lengths L_n of the sum of the sides are given by

$$S_n = \frac{na^2}{2}\sin\frac{2\pi}{n} \tag{6}$$

and

$$L_n = 2na\sin\frac{\pi}{n}, \tag{7}$$

respectively. For $n \geq 3$, the results are clear.

For $n = 2$, we will consider two diameters that are the same. We can consider it as a generalized regular polygon inscribed in the disc as a degenerate case. Then, $S_2 = 0$ and $L_2 = 4a$, and the general formulas are valid.

Next, we will consider the case $n = 1$. Then the corresponding regular polygon is a just diameter of the disc. Then, $S_1 = 0$ and $L_1 = 0$ that will mean that any regular polygon inscribed in the disc may not be formed and so its area and length of the side are zero.

Now we will consider the case $n = 0$. Then, by the division by zero calculus, we obtain that $S_0 = \pi a^2$ and $L_0 = 2\pi a$. Note that they are the area and the length of the disc. How to understand the results? Imagine contrary n tending to infinity, then the corresponding regular polygons inscribed in the disc tend to the disc. Recall our new idea that the point at infinity is represented by 0. Therefore, the results say that $n = 0$ regular polygons are $n = \infty$ regular polygons inscribed in the disc in a sense and they are the disc. This is our interpretation of the theorem:

Theorem. $n = 0$ *regular polygons inscribed in a disc are the whole disc.*

In addition, note that each inner angle A_n of a general n regular polygon inscribed in a fixed disc with radius a is given by

$$A_n = \left(1 - \frac{2}{n}\right)\pi. \tag{8}$$

The circumstances are similar for n regular polygons circumscribed in the disc, because the corresponding data are given by

$$S_n = na^2 \tan\frac{\pi}{n}, \tag{9}$$

$$L_n = 2na \tan\frac{\pi}{n}, \tag{10}$$

and Eq. (8), respectively.

3. Our life figure

As an interesting figure which shows an interesting relation between 0 and infinity, we will consider a sector Δ_α on the complex $z = x + iy$ plane

$$\Delta_\alpha = \left\{ |\arg z| < \alpha; 0 < \alpha < \frac{\pi}{2} \right\}.$$

We will consider a disc inscribed in the sector Δ_α whose center $(k, 0)$ with radius r. Then, we have

$$r = k \sin\alpha. \tag{11}$$

Then, note that as k tends to zero, r tends to zero, meanwhile k tends to $+\infty$, r tends to $+\infty$. However, by our division by zero calculus, we see that immediately that

$$[r]_{r=\infty} = 0. \tag{12}$$

For this fact, note the following:

The behavior of the space around the point at infinity may be considered by that of the origin by the linear transform $W = 1/z$. We thus see that

$$\lim_{z\to\infty} z = \infty, \tag{13}$$

however,

$$[z]_{z=\infty} = 0, \tag{14}$$

by the division by zero. Here, $[z]_{z=\infty}$ denotes the value of the function $W = z$ at the topological point at the infinity in one point compactification by Aleksandrov. The difference of Eq. (13) and Eq. (14) is very important as we see clearly by the function $W = 1/z$ and the behavior at the origin. The limiting value to the origin and the value at the origin are different. For surprising results, we will state the property in the real space as follows:

$$\lim_{x\to+\infty} x = +\infty, \quad \lim_{x\to-\infty} x = -\infty, \tag{15}$$

however,

$$[x]_{+\infty} = 0, \quad [x]_{-\infty} = 0. \tag{16}$$

Of course, two points $+\infty$ and $-\infty$ are the same point as the point at infinity. However, $\pm$ will be convenient in order to show the approach directions. In [8], we gave many examples for this property.

In particular, in $z \to \infty$ in Eq. (13), ∞ represents the topological point on the Riemann sphere, meanwhile ∞ in the left hand side in Eq. (13) represents the limit by means of the ϵ-δ logic.

On the sector, we see that from the origin as the point 0, the inscribed discs are increasing endlessly, however their final disc reduces to the origin suddenly - it seems that the whole process looks like our life in the viewpoint of our initial and final.

4. H. Okumura's example

The surprising example by H. Okumura will show a new phenomenon at the point at infinity.

On the sector Δ_α, we shall change the angle and we consider a fixed circle $C_a, a > 0$ with radius a inscribed in the sectors. We see that when the circle tends to $+\infty$, the angles α tend to zero. How will be the case $\alpha = 0$? Then, we will not be able to see the position of the circle. Surprisingly enough, then C_a is the circle with center at the origin 0. This result is derived from the division by zero calculus for the formula

$$k = \frac{a}{\sin\alpha}. \tag{17}$$

The two lines $\arg z = \alpha$ and $\arg z = -\alpha$ were tangential lines of the circle C_a and now they are the positive real line. The gradient of the positive real line is of course zero. Note here that the gradient of the positive imaginary line is zero by the division by zero calculus that means $\tan\frac{\pi}{2} = 0$. Therefore, we can understand that the positive real line is still a tangential line of the circle C_a.

This will show some great relation between zero and infinity. We can see some mysterious property around the point at infinity.

5. The point at infinity is represented by zero

By considering a line or the stereographic projection of a line, we will see that the point at infinity is represented by zero.

We write a line by the polar coordinate

$$r = \frac{d}{\cos(\theta - \alpha)}, \tag{18}$$

where $d = \overline{OH} > 0$ is the distance of the origin O and the line such that OH and the line is orthogonal and H is on the line, α is the angle of the line OH and the positive x axis, and θ is the angle OP ($P = (r, \theta)$ on the line) and the positive x axis. Then, if $\theta - \alpha = \pi / 2$: that is, OP and the line is parallel and P is the point at infinity, then we see that $r = 0$ by the division by zero calculus; the point at infinity is represented by zero and we can consider that the line passes the origin, however, it is in a discontinuous way.

That is, a line is, indeed, contains the origin; the true line should be considered as the sum of a usual line and the origin. We can say that it is a compactification of the line and the compacted point is the point at infinity, however, it is represented by $z = 0$.

The similar property of a line passing the origin may be looked by using a Hesse representation of a line.

Meanwhile, the envelop of the linear lines represented by, for constants m and a fixed constant $p > 0$,

$$y = mx + \frac{p}{m}, \tag{19}$$

we have the function, by using an elementary ordinary differential equation,

$$y^2 = 4px. \tag{20}$$

The origin of this parabolic function is missing from the envelope of the linear functions, because the linear equations do not contain the y axis as the tangential line of the parabolic function. Now recall that, by the division by zero, as the linear equation for $m = 0$, we have the function $y = 0$, the x axis. Note that both the x axis $y = 0$ and the parabolic function have the zero gradient at the origin; that will mean that in the reasonable sense the x axis is a tangential line of the parabolic function. Anyhow, by the division by zero, the envelope of the linear functions may be considered as the whole parabolic function containing the origin.

When we consider the limiting of the linear equations as $m \to 0$, we will think that the limit function is a parallel line to the x axis through the point at infinity. Since the point at infinity is represented by zero, it will become the x axis.

Meanwhile, when we consider the limiting function as $m \to \infty$, we have the y axis $x = 0$ and this function is an ordinary tangential line of the parabolic function. From these two tangential lines, we see that the origin has double natures; one is the continuous tangential line $x = 0$ and the second is the discontinuous tangential line $y = 0$.

6. A contradiction of classical idea for $1/0 = \infty$

As stated in Section 3, the infinity ∞ may be considered by the idea of the limiting, however, we had considered it as a number, for sometimes, typically, the point at infinity was represented by ∞ for some long years. For this fact, we will show a formal contradiction.

We will consider the stereographic projection by means of the unit sphere

$$\xi^2 + \eta^2 + \left(\zeta - \frac{1}{2}\right)^2 = 1$$

from the complex $z = x + iy$ plane. Then, we obtain the correspondences

$$x = \frac{\xi}{1-\zeta}, \quad y = \frac{\eta}{1-\zeta}$$

and

$$\xi = \frac{1}{2}\frac{z+\overline{z}}{z\overline{z}+1}, \quad \eta = \frac{1}{2i}\frac{z-\overline{z}}{z\overline{z}+1}, \quad \zeta = \frac{z\overline{z}}{z\overline{z}+1}.$$

In general, two points P and Q_1 on the diameter of the unit sphere correspond to z and z_1 respectively if and only if

$$z\overline{z}_1 + 1 = 0. \tag{21}$$

Meanwhile, two points P and Q_2 on the symmetric points on the unit sphere with respect to the plane $\zeta = \frac{1}{2}$ correspond to z and z_2 if and only if

$$z\overline{z}_2 - 1 = 0. \tag{22}$$

If the point P is the origin or the north pole, then the points Q_1 and Q_2 are the same point. Then, the identities Eq. (21) and Eq. (22) are not valid that show a contradiction.

Meanwhile, if we write Eq. (21) and Eq. (22)

$$z = -\frac{1}{\overline{z}_1} \tag{23}$$

and

$$z = \frac{1}{\overline{z}_2}, \tag{24}$$

respectively, we see that the division by zero Eq. (2) is valid.

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