Correctness Proof of Min-plus Algebra for Network Shortest-Paths Simultaneous Calculation

Hitapriya Suprayitno

Civil Engineering Dept., Institut Teknologi Sepuluh Nopember (ITS), Surabaya, INDONESIA.

suprayitno.hita@gmail.com, hitapriya@ce.its.ac.id

Keywords: transportation network analysis, special matrix technique, min-plus algebra, network shortest paths simultaneous calculation.

Abstract. Urban mass transit network need to be planned and evaluated. “Network quality” is one of the main mass transit qualities, in addition to operational quality. Connectivity and accessibility qualities are two important aspects of a “network quality”. This kind of analysis requires the whole point-to-point shortest-path data. A special matrix method has ever been developed for the simultaneous calculation of all the shortest-paths, for which a mathematical proof must be developed. To be mathematically correct, the calculation method must satisfy the following algorithm requirements : all itinerary combinations must be evaluated, each minimum result must be recorded, an appropriate fathoming procedure must exist, and the absolute minimum value must be produced automatically at the end of the calculation. The study proved that the min-plus algebra is capable of ensuring those procedures and always providing the correct answer. Thus, the min-plus algebra is considered as mathematically correct for network shortest-path simultaneous calculation.

1. Introduction

Big cities in Indonesia are considered as late in providing urban public mass transport. Therefore, a good mass transport network planning and evaluation are important. The first step to develop an urban mass transit is the development of a master plan, followed by the development of a step-by-step mass transit line. The “network quality” accessment is easily considered as capital. A “network quality” analysis requires considerably amount of network calculation. One of the essential data needed involves all node-to-node shortest path lengths [1-6].

Several methods have been developed for calculating the shortest-path. In operations research and discrete mathematics, the shortest-path can be calculated using the Dijkstra algorithm up to the Floyd algorithm for simultaneous calculation, by passing, among others, through the Kruskal-Wallis algorithm [7-10]. However, all these are still based on a list of link’s data. Moreover, the method is available in the graph theory [11]. In geography of transportation, an attempt was made to develop a method based on a special matrix calculation method; however, its capability is still limited and the matrix calculation is still not so practical [1]. All these methods are based on link data list; its practicality can still be improved.

Spreadsheet is a very powerfull and yet very versatile software for executing calculations. The most known and used spreadsheet is the Microsoft Excel, which is very excellent for tabular calculations, and of course any type of matrix calculations.

Sophisticated softwares for transportation modelling can be found easily; however these are not always easily accesible for financial reasons [12,13]. Operations research softwares are the same, but are not specially developed for transportation cases [7-9].

Hence, why do we not develop a special network calculation method designated to be easily used on a spreadsheet software and also to be easily written in programming language ?. Fortunately, a special matrix technique (SMT) for transportation network analyse (TNA) has been developed. This is specially intended, designed and developed to be easily used for spreadsheet software [3-6,14-16]. The classic matrix theory can not be used for this purpose [6,15,17].

One of the major matrix calculation technic is the min-plus algebra for all shortest-paths simultaneous calculation [6,14]. The min-plus algebra is a development of the max-plus algebra, which was first developed by INRIA in France to solve the discrete event system problems. Later, tentatives to use the max-plus algebra for transportation problem has ever been made [18-21]. After its initiation, the min-plus algebra has been used instead and always gave the correct result [6,14-16]. However, the proof of mathematical correctness is still not yet well developed and written.

This paper present the result of the mathematical proof of the correctness of the min-plus algebra for the all network's shortest-paths simultaneous calculation.

2. Special Matrix Technique for Transportation Network Analysis

2.1 SMT for TNA

The SMT for TNA was first developed for the sparse road network analysis. Later, it was developed and tested to be used for other types of TNA. The technique consists of : network case statements, network models, matrix form convention, and matrix calculation methods; the min-plus algebra is also its constituent [6,14-16].

2.2 Network Model

For the purpose of mathematical proof, a case of a road network was taken. The network model is a simple link-node model [6,14,16], in which the node can represent an intersection, or a region's important point, while the link represents a road segment. Fig. 1 illustrates a network model.

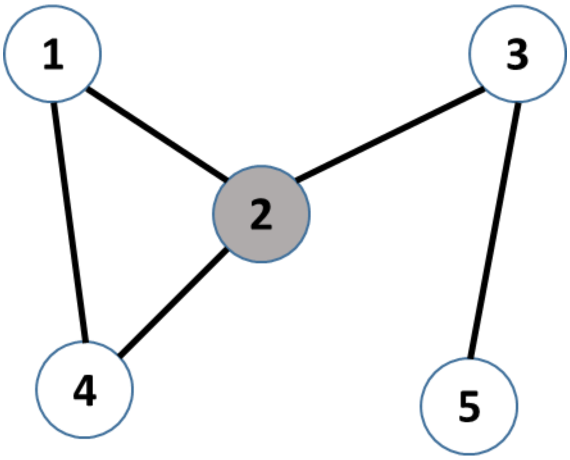

Fig. 1. Road network model

2.3 Matrix Representation of the Network

The matrix representation of a network is organized as follows : basic matrix, expanded matrix, and indicative matrix. A network is always represented as an **n×n** special matrix. The diagonal cells are used to represent node data, while link data must be represented in non diagonal cells [6,14,15]. Table 1 presents two samples of a network matrices.

Table 1. m.N and m.L matrices

m.N	1	2	3	4	5
1	1	0	0	0	0
2	0	3	0	0	0
3	0	0	1	0	0
4	0	0	0	1	0
5	0	0	0	0	1

m.L	1	2	3	4	5
1	0	1	0	1	0
2	1	0	1	1	0
3	0	1	0	0	1
4	1	1	0	0	0
5	0	0	1	0	0

2.4 Min-plus Algebra

The min-plus algebra is a slight development of the max-plus algebra, in which the "max operation" is changed into the "min operation". The general operations and notations in the min-plus algebra are presented as follows [6,14].

$$a \underline{\oplus} b = \min (a, b) \quad (1)$$
$$a \underline{\otimes} b = a + b \quad (2)$$
$$|A| \underline{\otimes} |B| = |C| \quad (3)$$
$$|C| = \underline{\bigoplus}_{k=1-n} a_{ik} \underline{\otimes} b_{kj} \quad (4)$$
$$c_{ij} = a_{i1} \underline{\otimes} b_{1j} \underline{\oplus} a_{i2} \underline{\otimes} b_{2j} \underline{\oplus} a_{i3} \underline{\otimes} b_{3j} \underline{\oplus} \ldots\ldots \underline{\oplus} a_{in} \underline{\otimes} b_{nj} \quad (5)$$
$$= \min ((a_{i1} + b_{1j}), (a_{i2} + b_{2j}), (a_{i3} + b_{3j}), \ldots.., (a_{in} + b_{nj})) \quad (6)$$
$$|A|^{\underline{\otimes} N} = |A| \underline{\otimes} |A|^{\underline{\otimes}(N-1)} \quad (7)$$
$$|A|^{\underline{\otimes} 0} = 1 \quad (8)$$

2.5 Shortest Path Formula in SMT for TNA

By referring to the SMT convention, the shortest-paths simultaneous calculation formulae is written as follows, by using the min-plus algebra principal [6,14].

$$m.SP = m.LL^{\underline{\otimes}(N-1)} \quad (9)$$

The power of (N-1) is utilized, because for a network with N nodes, for each pair of nodes; the maximum number of nodes can be passed without passing a node more than once, in traversing the network from node o (origin) to node d (destination), is N. Therefore, (N-1) is the maximum number of links that can be passed or the maximum number of steps [6,14].

3. Correctness Proof

3.1 Proof Method Development

Regarding the operations research, the shortest path calculation is an integer case optimization method. Several related algorithms have been developed, such as : enumeration of total combinations for shortest path calculation, general branch-and-bound technics for integer optimization and ordered combinations with a reduced searching area for general integer case [7-9,22,23]. All these algorithms deal with : the champ of investigation (the whole possible combination down to feasible solution champ), combination process and fathoming rule [7-9,22,23].

The mathematical proof is to verify whether the min-plus algebra calculation procedure satisfies the algorithm requirements.

The shortest path problem is an optimization case, in which a mixture of enumeration procedures and branch-and-bound technic used. Hence, the algorithm requirements to be fulfilled are as follows : all itinerary combinations must be evaluated, each minimum result must be recorded, correct and appropriate fathoming technic must be used, each absolute minimum must be automatically presented as a calculation result.

3.2 Correctness Proof

The proof must be produced by evaluating whether, the min-plus algebra for shortest-paths simultanous calculation satisfies the aforementioned algorithm requirement.

General Case. To be easily understandable, a general network case, consisting of 5 nodes and 8 links, is used to present the mathematical proof. The network and its link length matrices are presented in Figure 2 and Table 2. The value 0 in the table indicates the distance between node *i* to node *i*, all these can be found in the diagonal matrix cells. The value of 999 represents infinitive value, indicating that node *i* to node *j* is not connected directly.

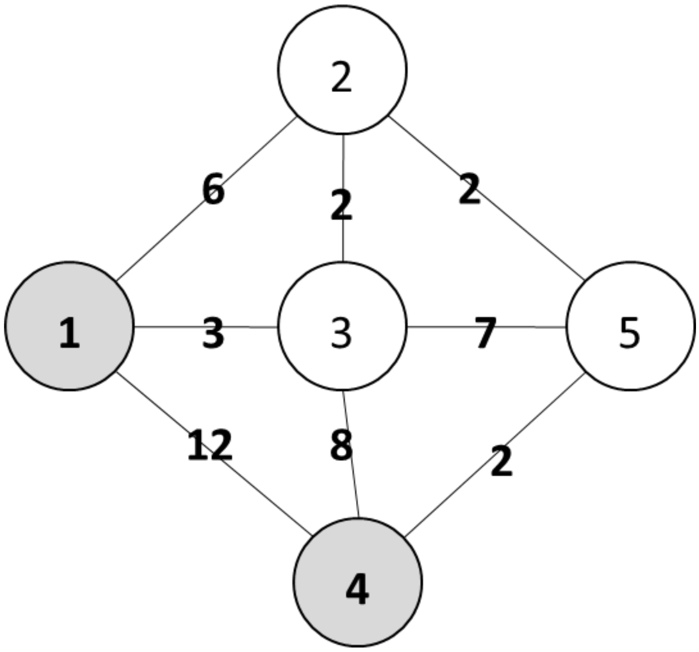

Fig. 2. General network case

Table 2. Link length matrix

m.LL	1	2	3	4	5
1	0	6	3	12	999
2	6	0	2	999	2
3	3	2	0	8	7
4	12	999	8	0	2
5	999	2	7	2	0

Shortest Path Calculation. The most important, pertinent and interesting part of the min-plus algebra matrix powering is the operation of matrix multiplication, which incorporate the following formulae :

$$ll^{n+1}_{ij} = \min \{(ll^{n}_{i1}+ll^{n}_{1j}),(ll^{n}_{i2}+ll^{n}_{2j}),(ll^{n}_{i3}+ll^{n}_{3j}) \ldots\ldots (ll^{n}_{in}+ll^{n}_{nj})\} \quad (10)$$

where :

ll^{n}_{ij} : the shortest path length from node *i* to *j*, for the n^{th} powering step

This operation ensures that all itinerary combinations are evaluated to obtain the itinerary combination with the minimum length. The operation is performed stepwise sequentially, according to the matrix powering step.

The min-plus algebra calculation, performed using a spread sheet, is presented in Figure 3 and Table 3. The figure 3 illustrates the entire itinerary combinations evaluation range.

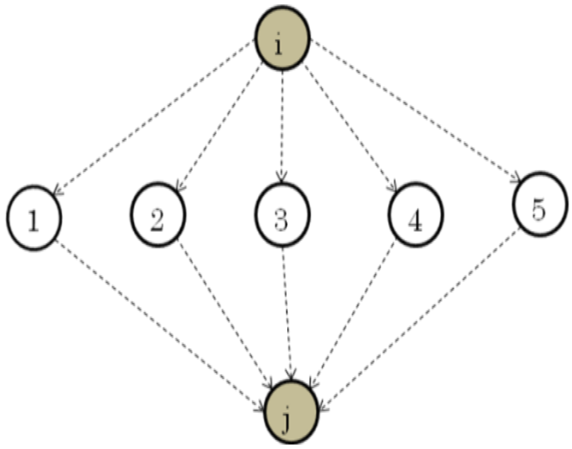

Fig. 3. Illustration of min-plus algebra power procedure

Table 3. Min-plus algebra calculation table

m.LL$^{\otimes 1}$	**1**	**2**	**3**	**4**	**5**
1	0	6	3	12	999
2	6	0	2	999	2
3	3	2	0	8	7
4	12	999	8	0	2
5	999	2	7	2	0

m.LL$^{\otimes 2}$	**1**	**2**	**3**	**4**	**5**
1	0	5	3	11	8
2	5	0	2	4	2
3	3	2	0	8	4
4	11	4	8	0	2
5	8	2	4	2	0

m.LL$^{\otimes 3}$	**1**	**2**	**3**	**4**	**5**
1	0	5	3	10	7
2	5	0	2	4	2
3	3	2	0	6	4
4	10	4	6	0	2
5	7	2	4	2	0

m.LL$^{\otimes 4}$	**1**	**2**	**3**	**4**	**5**
1	0	5	3	9	7
2	5	0	2	4	2
3	3	2	0	6	4
4	9	4	6	0	2
5	7	2	4	2	0

m.LL$^{\otimes 5}$	**1**	**2**	**3**	**4**	**5**
1	0	5	3	9	7
2	5	0	2	4	2
3	3	2	0	6	4
4	9	4	6	0	2
5	7	2	4	2	0

m.LL$^{\otimes 1}$	**1**	**2**	**3**	**4**	**5**
1	0	6	3	12	999
2	6	0	2	999	2
3	3	2	0	8	7
4	12	999	8	0	2
5	999	2	7	2	0

Calculation Detail and Connotation. For easy understanding, calculation detail and connotation explications are presented for the calculation of the shortest-path length from node 1 to 4 : $m.SP_{14}$. The discussion are detailed for calculations of Steps 0-4. For each step the results are expressed by $m.LL_{14}^{\underline{\otimes}1}$ to $m.LL_{14}^{\underline{\otimes}5}$.

Step 0. Step 0 represents power 1, $m.LL^{\underline{\otimes}1}$, indicating that one step itinerary must be derived for all pairs of origins and destinations.

$m.LL_{14}^{\underline{\otimes}1}$ $= m.LL_{14}$
$= ll_{14}$
$= 12$

Step 1. Step 1 represents power 2, $m.LL^{\underline{\otimes}2}$, implying that a two steps itinerary must be derived for all pairs of origins and destinations. The calculation for the shortest-path itinerary from node 1 to 4 is presented as follows.

$m.LL_{14}^{\underline{\otimes}2}$ $= m.LL_{14} \underline{\otimes} m.LL_{14}$
$= \min\{(ll_{11} + ll_{14}), (ll_{12} + ll_{24}), (\mathbf{ll_{13} + ll_{34}}), (ll_{14}+ll_{44}), (ll_{15} + ll_{54})\}$
$= \min\{(0 + 12),(6+999),(\mathbf{3+8}),(12+0),(999+2)\}$
$= 11$

All itineraries are tested as follows :

$ll_{11} – ll_{14}$: real path : 1-1-4, path length = 0 + 12 = 12.
$ll_{12} – ll_{24}$: real path : 1-2-4, path length = 6 + 999 = 999.
$\mathbf{ll_{13} – ll_{34}}$: real path : 1-3-4, path length = 3 + 8 = 11.
$ll_{14} – ll_{44}$: real path : 1-4-4, path length = 12 + 0 = 12.
$ll_{15} – ll_{54}$: real path : 1-5-4, path length = 999 + 2 = 999.

Step 2. Step 2 represents power 3, $m.LL^{\underline{\otimes}3}$, indicating that a three-steps itinerary must be derived for all pairs of origins and destinations. The calculation for the shortest-path itinerary from Node 1 to 4 is presented as follows.

$m.LL_{14}^{\underline{\otimes}3}$ $= m.LL^{\underline{\otimes}2}{}_{14} \underline{\otimes} m.LL_{14}$
$= \min\{(ll^2{}_{11} + ll_{14}), (ll^2{}_{12} + ll_{24}), (ll^2{}_{13} + ll_{34}), (ll^2{}_{14}+ll_{44}), (\mathbf{ll^2{}_{15} + ll_{54}})\}$
$= \min\{(0 + 12),(5+999),(3+8),(11+0),(\mathbf{8+2})\}$
$= 10$

Optimum itinerary :

$ll_{15} – ll_{54}$: $path^3$: 1-5-4, path length = 8 + 2 = 10.
$path^2 15$: 1-2-5
real path : 1-2-5-4

Step 3. Step 3 represents power 4, $m.LL^{\underline{\otimes}4}$, indicating that a four-steps itinerary must be derived for all pairs of origins and destinations. The calculation for the shortest-path itinerary from Node 1 to 4 is presented as follows.

$m.LL_{14}^{\underline{\otimes}4}$ $= m.LL^{\underline{\otimes}3}{}_{11} \underline{\otimes} m.LL_{14}$
$= \min\{(ll^3{}_{11} + ll_{14}), (ll^3{}_{12} + ll_{24}), (ll^3{}_{13} + ll_{34}), (ll^3{}_{14}+ll_{44}), (\mathbf{ll^3{}_{15} + ll_{54}})\}$
$= \min\{(0 +12),(5+999),(3+8),(10+0),(\mathbf{7+2})\}$
$= 9$

Optimum itinerary :

$ll_{15} – ll_{54}$: $path^4$: 1-5-4, path length = 7 + 2 = 9.
$path^3 15$: 1-2-5
$path^2$ 12 : 1-3-2
real path : 1-3-2-5-4

Step 4. Step 4 represents power 5, $m.LL^{\otimes 5}$, implying that a five steps itinerary is derived for all pairs of origins and destinations. The calculation for the shortest-path itinerary from Node 1 to 4 is presented as follows.

$m.LL_{14}^{\otimes 5}$ $= m.LL^{\otimes 4}_{14} \otimes m.LL_{14}$
$= \min \{(ll^4_{11} + ll_{14}), (ll^4_{12} + ll_{24}), (ll^4_{13} + ll_{34}), (\mathbf{ll^4_{14}+ll_{44}}), (\mathbf{ll^4_{15} + ll_{54}})\}$
$= \min \{(0 + 12),(5+999),(3+8),(\mathbf{9+0}),(\mathbf{7+2})\}$
$= 9$

Optimum itinerary :

$ll_{14} - ll_{44}$: path[5] : 1-4-4, path length = 9 + 0 = 9.
real path[4] 14 : 1-3-2-5-4
real path : 1-3-2-5-4-4

$ll_{15} - ll_{54}$: path[5] : 1-5-4, path length = 7 + 2 = 9.
real path[4] 15 : 1-3-2-5-5
real path : 1-3-2-5-5-4

Calculation Notes. Principal characteristics of min-plus algebra matrix power calculation can be drawn and presented as follows :

- In terms of network, each step of the powering process, implies the addition of one-step-forward in the shortest path itinerary searching and is calculated in terms of path length.
- For each one-step-forwards (each one-step-powerings), the minimum path length is always calculated and used, among the all existing combinations.
- The minimum value and optimum itinerary can then be different for every step.
- Once the minimum absolute is obtained, the value remains the same throughout the next powering.
- After all the cells are already filled with the minimum absolute value, each matrix cell values remains the same throughout the next powering ($m.LL^{\otimes 5} = m.LL^{\otimes 4}$).
- The total absolute minimum can be achieved at power (N-1) or earlier.

3.3 Satisfaction of Algorithm Requirements

The entire min-plus algebra power operation principal, that is *matrix to matrix multiplication mechanism*, for simultaneous shortest-path length calculation, was shown to satisfy all the algorithm requirements stated earlier.

- All possible itinerary combinations are verified.
- Only minimum result is recorded for each cell in each step.
- The fathoming procedure is realized by always taking the minimum value of each step's combination, thus incorporating the fact that the optimum itinerary for a cell can be different across different steps.
- After obtaining the absolute minimum values, they always remains the same even though the powering process continues.
- The absolute minimum is obtained automatically at the calculation end.

4. Conclusions

The study's objective was successfully achieved. The main conclusion points, of the min-plus algebra matrix powering, are presented as follows.

- It can be used for the simultaneous calculation of the shortest-path length and is considered mathematically correct.
- It can be very easily performed on spreadsheet type software and can be easily programmed through a programming language.
- It is not capable of indicating the shortest path itineraries directly.

Further researches should be conducted to develop a matrix calculation method to automatically record the shortest path itineraries and investigate the possibilities of using the min-plus algebra for urban mass transit schedule synchronization.

References

[1] J.E. Taaffe, H.L. Gauthier & M.E. O'Kelly, "Geography of Transportation", Second Edition, ***Prentice Hall***, New Jersey, 1996.

[2] Ofyar Z. Tamin, "Perencanaan, Pemodelan dan Rekayasa Transportasi: Teori, Contoh Soal dan Aplikasi", Edisi ke 2, ***Penerbit ITB***, Bandung, 2008.

[3] H. Suprayitno, I.B. Mochtar, & A. Wicaksono, "Tree Graph Non Existence Concept for Non Directed Non Weighted Graph", ***ISSTEC 2009 – International Seminar on Science and Technology***, (Universitas Islam Indonesia, Yogyakarta), 24 January 2009.

[4] Hitapriya Suprayitno, "Road Network Quality Components, Case of Rural Road Network with Low Traffics". ***APTE 2010 Seminar,*** (Diponegoro University. Semarang), 12 June 2010.

[5] H. Suprayitno, I.B. Mochtar & A. Wicaksono, "Assessment to A Max-Plus Algebra Power Operation on UnWeighted Transportation Network Model of Its Behavior, Connotation, and Utilization", ***Prosiding Seminar Nasional Teknik Sipil X,*** (Jurusan Teknik Sipil, ITS, Surabaya), 2014.

[6] Hitapriya Suprayitno, "Metoda Penilaian Kualitas Jaringan Jalan Utama – di Wilayah Kabupaten", ***Disertasi RC 09-3399***, Jurusan Teknik Sipil, Fakultas Teknik Sipil dan Perencanaan. Institut Teknologi Sepuluh November, Surabaya, 2014.

[7] F.S. Hillier & G.J. Lieberman, "Introduction to Operations Research", Fifth Edition, ***Holden-Day Inc.***, San Francisco, 1990.

[8] Hamdy A. Taha, "Operations Research – An Introduction", Fifth Edition, ***Maxwell Macmillan International***, New York, 1992.

[9] A.W. Goldberg, "Basic Shortest Path Algorithms", ***DIKU Summer School on Shortest Paths***, Microsoft Research, Silicon Valey, 2000.

[10] Rinaldi Munir, "Matematika Diskrit", Edisi Kelima, ***Penerbit Informatika***, Bandung, 2012.

[11] J.A. Bondy & U.S.R. Murty, "Graph Theory with Applications", Fifth Printing, ***North-Holland***, New York, 1982.

[12] INRO, "EMME/2 User's Manual – Software Release 9", ***INRO Consultant Inc.***, Canada, 1999.

[13] PTV Group, "VISSUM 15 User's Manual", ***PTV Group***. Karlsuhre. 2016.

[14] H. Suprayitno, I.B. Mochtar, & A. Wicaksono, "A Special Matrix Power Operation Development for Simultaneous Calculation of All Network's Shortest Path", ***JATIT - Journal of Applied and Theoritical Information Technology***, 10 April 2014, Vol. 62, No. 1, 2014.

[15] Hitapriya Suprayitno, "Development of A Special Matrix Technics for Road Network Analysis. Case of Identifying Un-Connected and Miss-Connected Node", ***18th FSTPT International Symposium,*** (Lampung University, Bandar Lampung), 27-30 August 2015.

[16] Hitapriya Suprayitno, "In Search of Basic Principles for Transportation Network Analysis", ***Seminar Nasional FSTPT '16***, (Universitas Syah Kuala, Banda Aceh), 2016.

[17] Serre Denis, "Matrices : Theory and Applications", ***Springer-Verlag***, New York, 2002.

[18] F. Baccelli, G. Cohen, G.J. Olsder & J-P Quadrat, "Synchronization and Linearity – An Algebra for Discrete Event Systems", ***John Wiley & Son***, New York, 1992.

[19] B. de Schutter & T. van de Boom, "Max-plus Algebra and Max-Plus Linear Discrete Event Systems : An Introduction", ***Proceeding of 9th International Workshop on Discrete Event Systems (WODES '08)***, (Goteborg), 2008.

[20] R. de Vries, B. de Schutter & B. de Moor, "On Max-Algebraic Models for Transportation Networks", ***Proceeding of 4th International Workshop on Discrete Event Systems (WODES '98)***, (Cagliari), 1998.

[21] Hitapriya Suprayitno, "Identifikasi Graf Pohon Berbasis Aljabar Maks-Plus – Pada Graf Tak Berarah Tak Berbobot", ***Seminar Nasional Teknik Sipil VI,*** (Jurusan Teknik Sipil, ITS, Surabaya), 27 Januari 2010.

[22] Hitapriya Suprayitno, "Penyusunan Algoritma bagi Pemecahan Permasalahan Optimasi Linier Umum Bersolusi Bilangan Bulat", ***Jurnal TORSI***, (Jurusan Teknik Sipil, ITS, Surabaya), Tahun ke 19 No. 1, Edisi Mei 1999.

[23] Hitapriya Suprayitno, "Metoda Enumerasi Total bagi Perhitungan Lintasan Terpendek Antara Dua Titik", ***Seminar Nasional Teknik Sipil IV,*** (Jurusan Teknik Sipil, ITS, Surabaya), 13 Februari 2008.