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Propagation of Stress Waves in Viscoelastic Rods and Plates (2) - Acoustic Impedance of a Viscoelastic Rod: Validation with Literature Data -

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Abstract. We derived the explicit analytical expression of the complex wave number of a longitudinal wave in a viscoelastic rod and a flexural wave in a viscoelastic beam. This paper proposes a method for obtaining the acoustic impedance from a complex wave number for stress waves in viscoelastic rods. This acoustic impedance is a frequency response function (FRF) in the form of a ratio of acoustic pressure to particle velocity. The complex wave number is related to the viscoelasticity through the formula presented in our previous paper. Calculation results were validated via a comparison with literature data. We first converted the acoustic impedance into an FRF in the form of the ratio of the displacement at the edge to the excitation force. Subsequently, this FRF was compared with the results of cork from the study by Policarpo et al. Our results were consistent with their results.

1. Introduction

Lightweight and soft materials are used as shock-absorbing and sound-absorbing materials. Among these, breathable foamed resins (open foam) such as polyurethane and melamine and fiber materials such as glass wool and wool are used as sound-absorbing materials. In the case of these materials, it is important to model the interaction between the waves propagating in the skeleton and in the fluid [1].

It will be easier to model the propagation of elastic waves in shock-absorbing or sound-absorbing materials acting as a spatially homogeneous linear viscoelastic body. Filippov et al. [2] theoretically investigated the propagation of compressive waves in a semi-infinite viscoelastic rod. Musa [3] calculated wave propagation in a viscoelastic rod with both ends sandwiched between semi-infinite elastic rods. He used the three-parameter Maxwell model as a constitutive equation and solved the Laplace transformed equation of motion.

Policarpo et al. [4] and Sasso et al. [5] studied the propagation of elastic waves in a cork rod. According to Sasso et al. [5], cork behaves as a viscoelastic material without breathability. The Poisson's ratio of the cork is almost zero, showing no lateral displacement with respect to the load. Cork has durability in addition to excellent sound and shock absorption. Policarpo et al. [4] hammer-excited a cork specimen sandwiched between steel rods on both sides. They identified the storage modulus and loss factor of the cork specimen for each eigenmode from the frequency response function (FRF). Sasso et al. [5] measured the wave-based FRF using split Hopkinson pressure bar [6]. They also measured large nonlinear distortion using image processing.

We have studied stress waves propagating in viscoelastic rods, beams, and plates [7]. In the previous paper [7], we showed explicit formulas expressing the complex wave number (real wave number and attenuation constant) for given viscoelasticity (storage modulus and loss factor). In the case of a longitudinal wave of a plane wave propagating in a viscoelastic rod, compressive stress and velocity at each position can be regarded as acoustic pressure and particle velocity, respectively. Therefore, the acoustic impedance can be used as the FRF of the viscoelastic rod as in the case of an acoustic tube. However, before referring to [4] and [5], we did not know what kind of shock-absorbing or sound-absorbing material behaves as a spatially homogeneous viscoelastic material.

The aim of this study is to verify the acoustic impedance using our formulas of the complex wave number via a comparison with literature data.

2. Frequency Response of a Viscoelastic Rod by using Acoustic Impedance

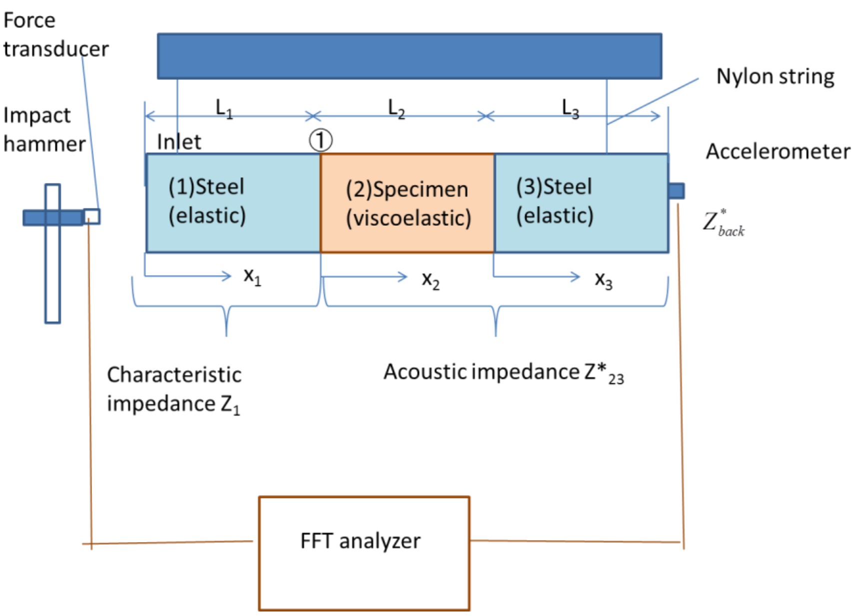


Fig. 1. Experimental setup of Policarpo et al. [5]

Policarpo et al. [4] measured the FRF (acceptance) in the form of (force) / (displacement), and they estimated the viscoelastic data using the modal-based inverse calculations. We calculated the complex wave number (real wave number and attenuation constant (factor)) using the viscoelastic data estimated by Policarpo et al. Subsequently, we calculated the acoustic impedance defined later from the complex wave number. Subsequently, we converted the acoustic impedance into the FRF (acceptance) and compared it with their experimental results.

2.1 Propagation of Stress Waves in Viscoelastic Rod

We deal with the longitudinal plane wave propagating in the viscoelastic rod. As described in the previous paper [7], the amplitude of the tensile stress and the displacement can be expressed as follows (For details, see Appendix A):

Tensile Stress

$$\sigma = E^*(\omega)\frac{\partial Y(\omega,x)}{\partial x}\exp(i\omega t), \qquad (2\_1)$$

where

$Y(\omega,x)$: Amplitude of the displacement [m],

$\sigma(t,x)$: Tensile stress [Pa], ω : radial frequency [rad/s], t : time [s] , x : location [m],

$$E^*(\omega) = E'(\omega) + iE''(\omega),\ i = \sqrt{-1}$$

($E^*(\omega)$: Complex modulus (Pa), $E'(\omega)$: storage modulus (Pa), $E''(\omega)$: loss modulus (Pa))

Response of the Displacement Amplitude

$$\omega^2 Y(\omega, x) + \frac{E^*(\omega)}{\rho}\frac{\partial^2 Y(\omega, x)}{\partial x^2} = 0 \qquad (2\_2)$$

$$Y(\omega, x) = B_1 \cdot \exp\left[-i\left(\rho/E^*(\omega)\right)^{\frac{1}{2}}\omega x\right] + B_2 \cdot \exp\left[i\left(\rho/E^*(\omega)\right)^{\frac{1}{2}}\omega x\right],\ i = \sqrt{-1} \qquad (2\_3)$$

B_1, B_2: Constants, ρ: mass density [kg/m^3]

2.2 Acoustic Pressure and Particle Velocity in a Viscoelastic Rod

For the longitudinal wave of radial frequency ω, the displacement X at position x and time t is expressed as follows:

$$X(\omega, t, x) = B_1 \cdot \exp\left[i\omega\left\{t - \left(\rho/E^*(\omega)\right)^{\frac{1}{2}} x\right\}\right] + B_2 \cdot \exp\left[i\omega\left\{t + \left(\rho/E^*(\omega)\right)^{\frac{1}{2}} x\right\}\right] \qquad (2\_4)$$

X : Displacement [m].

For this displacement, the velocity $u = \partial X/\partial t$ at the position x is regarded as the particle velocity, and the compressive stress $p = -E^*(\omega)\partial X/\partial x$ is regarded as the acoustic pressure. Here, u and p can be expressed as follows:

Particle Velocity [m/s]

$$u = \frac{\partial X}{\partial t} = i\omega B_1 \cdot \exp\left[i\omega\left\{t - \left(\rho/E^*(\omega)\right)^{\frac{1}{2}} x\right\}\right] + i\omega \cdot B_2 \cdot \exp\left[i\omega\left\{t + \left(\rho/E^*(\omega)\right)^{\frac{1}{2}} x\right\}\right] \qquad (2\_5)$$

Acoustic Pressure [Pa]

$$\begin{aligned} p &= -E^*(\omega)\frac{\partial X}{\partial x} \\ &= iB_1\omega\left[\rho E^*(\omega)\right]^{\frac{1}{2}} \cdot \exp\left[i\omega\left\{t - \left(\rho/E^*(\omega)\right)^{\frac{1}{2}} x\right\}\right] - iB_2\omega\left[\rho E^*(\omega)\right]^{\frac{1}{2}} \cdot \exp\left[i\omega\left\{t + \left(\rho/E^*(\omega)\right)^{\frac{1}{2}} x\right\}\right] \end{aligned} \qquad (2\_6)$$

2.3 Characteristic Impedance

If the constant B_2 or B_1 is 0, Eq. (2_5) and Eq. (2_6) represent longitudinal waves propagating in the positive and negative x directions, respectively. By using Eqs. (A2_6a,b) in Appendix A,

(When B_2=0) a forward-going wave propagating while attenuating toward the positive x direction

$$p = \left[\rho E^*(\omega)\right]^{1/2} u \qquad (2\_7a)$$

(When B_1=0) a backward-going wave propagating while attenuating toward the negative x direction

$$p = -\left[\rho E^*(\omega)\right]^{1/2} u \qquad (2\_7b)$$

We define the ratio of the acoustic pressure and the particle velocity expressed by Eq. (2_7 a, b) as the characteristic impedance Z_C^*. Thus,

(Complex) Characteristic Impedance [$Pa \cdot s/m$]

$$Z_C^* := \left[\rho E^*(\omega)\right]^{1/2}. \qquad (2\_8)$$

This is a similar definition to that for acoustic waves in air.

2.4 Relations between the Characteristic Impedance and the Complex Wave Number

The characteristic impedance of the viscoelastic rod (Eq. (2_8)) can be expressed as follows using the complex wave number (See Appendix A):

$$Z_c^*(\omega) = \mathrm{Re}(Z_c^*) + i\,\mathrm{Im}(Z_c^*),$$

where (2_9)

$$\mathrm{Re}(Z_c^*) = \frac{\rho\omega\beta}{\beta^2+\alpha^2},\ \mathrm{Im}(Z_c^*) = \frac{\rho\omega\alpha}{\beta^2+\alpha^2},\ i=\sqrt{-1}$$

where

Real Wave Number [1/m]

$$\beta(\omega) = \omega\left[\frac{\rho}{2E'(\omega)}\right]^{\frac{1}{2}}\left[\frac{1}{(1+\tan^2\delta)^{1/2}} + \frac{1}{1+\tan^2\delta}\right]^{\frac{1}{2}} \quad (2\_10)$$

Attenuation Constant [1/m]

$$\alpha(\omega) = \omega\left[\frac{\rho}{2E'(\omega)}\right]^{\frac{1}{2}}\left[\frac{1}{(1+\tan^2\delta)^{1/2}} - \frac{1}{1+\tan^2\delta}\right]^{\frac{1}{2}}, \quad (2\_11)$$

where

$\tan\delta = E''(\omega)/E'(\omega)$: Loss tangent (loss factor) [-] (See [7])

Here, we denote the complex wave number as follows according to the notation with a sound-absorbing material:

Complex Wave Number [1/m]

$$k_C^* = \beta - i\alpha \quad (2\_12)$$

2.5 Acoustic Impedance of the Series Rods

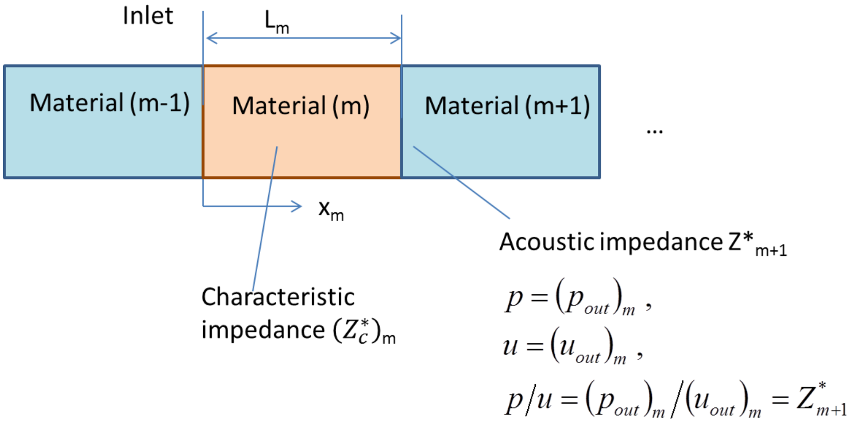


Fig.2 Series connection of rods

The propagation of the longitudinal plane wave in the viscoelastic rod can be expressed by the same form of equations as the propagation of the acoustic wave in the acoustic tube.

[Acoustic Impedance at the Entrance of the material (m)]

In Fig. 2, the acoustic impedance at the left end of the material (m) can be expressed as a ratio of the acoustic pressure and the particle velocity as follows (See Appendix B):

$$Z_m^* := (p)_{xm=0} / (u)_{xm=0} = (Z_C^*)_m \frac{(Z_c^*)_m \sinh(ik_m^* L_m) + Z_{m+1}^* \cosh(ik_m^* L_m)}{(Z_c^*)_m \cosh(ik_m^* L_m) + Z_{m+1}^* \sinh(ik_m^* L_m)} \quad (2\_13)$$

where

$(Z_c^*)_m$: Characteristic impedance of the material (m)[Pa・s/m],

Z_m^* : Acoustic impedance viewed from the left end of the material (m) [Pa・s/m],

$k_m^* = \beta_m - i\alpha_m$: Complex wave number in the material (m) [1/m],

β_m , α_m: Real wave number and attenuation constant in the material (m) [1/m].

Eq. (2_13) is in the form of a recurrence formula for section number m.

[Pressure Ratio and Velocity Ratio between Both Ends of the material (m)]

The following acoustic pressure ratio and particle velocity ratio are obtained from the distribution of acoustic pressure and particle velocity in the material (m).

$$\frac{(p)_{xm=0}}{(p)_{xm=Lm}} = \cosh(ik_m^* L_m) + \frac{(Z_c^*)_m}{Z_{m+1}^*} \sinh(ik_m^* L_m) \ , \ \frac{(u)_{xm=0}}{(u)_{xm=Lm}} = \cosh(ik_m^* L_m) + \frac{Z_{m+1}^*}{(Z_c^*)_m} \sinh(ik_m^* L_m) \quad (2\_14)$$

These ratios can be used to convert the acoustic impedance of the whole system into the FRF (acceptance).

2.6 Acoustic Impedance of the Whole System and the Specimen

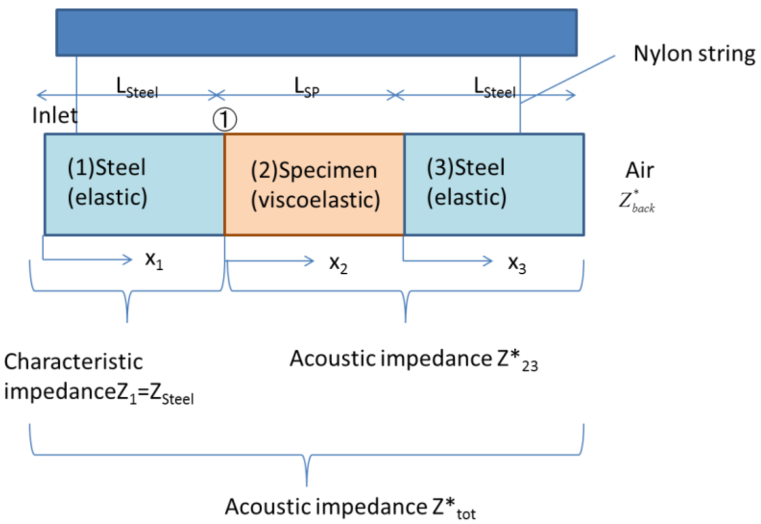


Fig.3. Acoustic impedance of the specimen Z_{23}^* and the whole system Z_{tot}^*

By applying the results in the previous section to the system in Fig. 3, ratio of the acoustic impedance of the whole system to the characteristic impedance of steel can be expressed as follows:

$$\frac{Z_{tot}^*}{Z_{Steel}} = \frac{(Z_{23}^*/Z_{Steel}) + i\tan(\omega L_{Steel}/c_{Steel})}{1 + i(Z_{23}^*/Z_{Steel})\tan(\omega L_{Steel}/c_{Steel})},$$

where (2_15)

$$Z_{Steel} = \rho_{Steel} c_{Steel} \ , \ c_{Steel} = \sqrt{\frac{E_{Steel}(1-\nu_{Steel})}{\rho_{Steel}(1+\nu_{Steel})(1-2\nu_{Steel})}}$$

Z_{Steel} : Characteristic impedance of the steel rod $[Pa \cdot s/m]$,
c_{Steel}: Phase velocity in steel [m/s], ρ_{Steel}: mass density of steel $[kg/m^3]$,
ν_{Steel}: Poisson's ratio of steel [-]

Here, Z_{23}^* is the acoustic impedance as viewed from the left end of the specimen expressed by Eq. (2 _16).

$$Z_{23}^* = Z_{SP}^* \cdot \frac{1 + \frac{Z_{SP}^*}{Z_3^*}\tanh[(\alpha + i\beta)L_{SP}]}{\frac{Z_{SP}^*}{Z_3^*} + \tanh[(\alpha + i\beta)L_{SP}]} = Z_{SP}^* \cdot \frac{\coth[(\alpha + i\beta)L_{SP}] + \frac{Z_{SP}^*}{Z_3^*}}{1 + \frac{Z_{SP}^*}{Z_3^*}\coth[(\alpha + i\beta)L_{SP}]},$$

where (2_16)

$$Z_{SP}^* = \frac{\rho\omega(\beta + i\alpha)}{\beta^2 + \alpha^2},$$

where
β and α : Real wave number and attenuation constant [1/m] (Eqs. (2_10) and (2_11).)

$$Z_3^* = Z_{3b}^* \approx iZ_{Steel}\tan(\omega L_{Steel}/c_{Steel}) \tag{2_17}$$

Acoustic impedance of the rightmost steel rod (See Appendix C)
Relations between hyperbolic functions

$$\coth[(\alpha + i\beta)L_{SP}] = [\sinh(2\alpha L_{SP}) - i\sin(2\beta L_{SP})]/[\cosh(2\alpha L_{SP}) - \cos(2\beta L_{SP})] \tag{2_18}$$

where
Z_{SP}^*: Characteristic impedance of the specimen $[Pa \cdot s/m]$,
Z_3^* : Acoustic impedance viewed from the left end of the rightmost steel rod $[Pa \cdot s/m]$,
L_{Steel} Length of the steel rod [m],
L_{SP}: Length of the specimen [m].

2.7 Conversion from the Acoustic Impedance of the Whole System to the FRF

Here, we consider a method to convert the acoustic impedance shown in the previous section into the FRF in the form of (displacement) / (force). First, (displacement) / (force) at the left end in Fig. 3 can be expressed using the acoustic impedance Z_{tot}^* as follows :

Acceptance

$$H_{x_1=0} = \left(\frac{X}{F}\right)_{x_1=0} = \frac{1}{Ap}\frac{u}{i\omega} = \frac{1}{i\omega A}\frac{u}{p} = \frac{1}{i\omega A Z_{tot}^*} \tag{2_19}$$

(X: amplitude of the displacement [m], F: amplitude of the excitation force [N], A: cross-sectional area $[m^2]$)
We regarded the acoustic impedance $Z_{tot}{}^*$ of the system divided by the following velocity ratio (Eq. (2_20)) as the FRF (acceptance) (See Eq. (2_21)).

Velocity Ratio between Both Ends of the Specimen

$$R_{at}U := (u_C)_{x_2=0}/(u_C)_{x_2=L_{SP}}$$

$$= \cos(\beta L_{SP})\cosh(\alpha L_{SP}) + i\sin(\beta L_{SP})\sinh(\alpha L_{SP}) + \frac{Z_3^*}{Z_{SP}^*}[\cos(\beta L_{SP})\sinh(\alpha L_{SP}) + i\sin(\beta L_{SP})\cosh(\alpha L_{SP})]$$

(2_20)

Acceptance

$$H_{Specimen} = (X)_{x_2=L_{SP}} / (F)_{x_1=0} = (X/F)_{x_1=0} \cdot \frac{(X)_{x_2=L_{SP}}}{(X)_{x_2=0}} = \frac{1}{i\omega A Z_{tot}^{*} \cdot R_{at} U} \qquad (2\_21)$$

(Influence of Velocity Ratio between Both Ends of the Steel Rod)

The velocity ratio between the two ends of the steel rod at the left end of Fig. 3 can be expressed using Eq. (B1 _ 10) in Appendix B as follows:

$$\frac{(u)_{x1=0}}{(u)_{x1=L_{steel}}} = cosh(i\,\omega L_{steel}/c_{steel}) + \frac{Z_{SP}^{*}}{Z_{steel}} sinh(i\,\omega L_{steel}/c_{steel}) \approx 1 \qquad (2\_22)$$

Therefore, the velocity ratio between the ends of the steel rod does not affect the FRF (acceptance).

3. Numerical Calculation and Comparison with Literature Data

The physical properties and dimensions used in our calculations are listed in Tables 1 and 2. The data in these tables are the values described in the paper presented by Policarpo et al. [4]. Here, the storage modulus and the loss factor were estimated by these authors via inverse analysis. The two frequencies correspond to the two natural frequencies obtained by changing the lengths of the steel rods at both ends. We used the value obtained via linear interpolation of the data listed in the tables as a relation between viscoelasticity and frequency.

The mass density and the storage modulus of specimen A are higher than those of specimen B. The loss factor of specimen B is higher than that of specimen A. Thus, specimen A is rather heavy and hard and has low viscosity. Specimen B, in contrast, is light, soft, and highly viscous.

Table 1.Properties and size of the specimen (Specimen A)

[Conditions]			
(Steel)			
ρ_{Steel}	Mass density [kg/m^3]	7640	
E_{Steel}	Young's modulus [GPa]	205	
ν_{Steel}	Poisson's ratio [−]	0	(*1)
c_{Steel}	Phase velocity [m/s]	5180.0	
Z_{Steel}	Characteristic impedance [Pa・s/m]	3.958E+07	
(Specimen)	Cork A		(*2)
ρ	Mass density [kg/m^3]	893.0	
(Viscoelasticity vs. frequency)			(*3)
	E'	$tan\delta$	
Frequency (Hz)	Storage modulus [MPa]	Loss factor [−]	
159.7	48.8	0.155	
1124.0	49.5	0.187	
[Dimension]			
(Length)			
Steel	L_{Steel} [mm]	20.5	
Specimen	L_{SP} [mm]	12.8	
Total	$L_{tot} = L_{SP} + 2L_{Steel}$ [mm]	53.8	
(Cross sectional area)			
$A_{rea}[m^2]$	4.0E−03		

(*1) Varied in the range $0 \leq \nu_{Steel} < 0.5$
(*2) and (*3): Policarpo's data (for their VC6400, (*3): Policarpo's modal inversion results

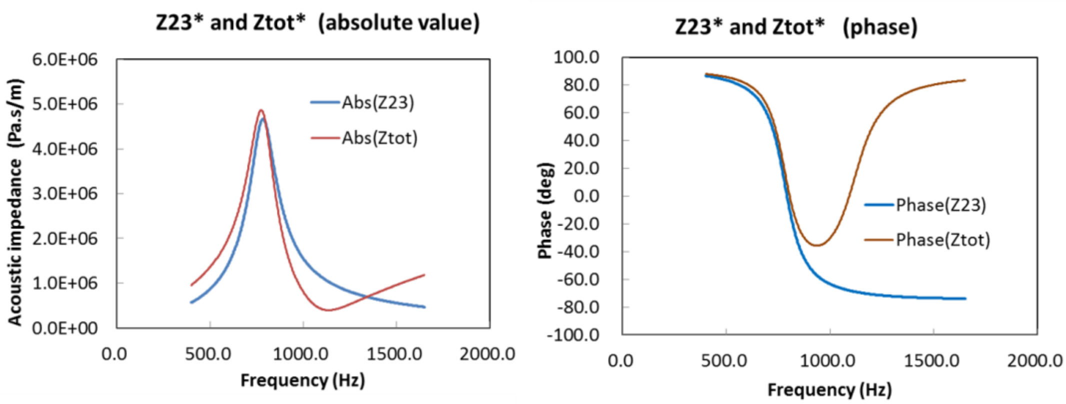
Table.2 Properties and size of the specimen (Specimen B)

[Conditions]			
(Steel)			
ρ_{Steel}	Mass density [kg/m^3]	7640	
E_{Steel}	Young's modulus [GPa]	205	
ν_{Steel}	Poisson's ratio [−]	0.4	(*1)
c_{Steel}	Phase velocity [m/s]	7582.8	
Z_{Steel}	Characteristic impedance [Pa s/m]	5.793E+07	
(Specimen)	Cork B		(*2)
ρ	Mass density [kg/m^3]	516.0	
(Viscoelasticity vs. frequency)			(*3)
	E'	$tan\delta$	
Frequency (Hz)	Storage modulus [MPa]	Loss factor [−]	
44.9	2.9	0.225	
319.1	3.1	0.245	
[Dimension]			
(Length)			
Steel	L_{Steel} [mm]	20.7	
Specimen	L_{SP} [mm]	10.1	
Total	$L_{tot} = L_{SP} + 2L_{Steel}$ [mm]	51.5	
(Cross sectional area)			
$A_{rea}[m^2]$	4.0E−03		

(*1) Varied in the range $0 \leq \nu_{Steel} < 0.5$
(*2) and (*3): Policarpo's data (for their VC1100, (*3): Policarpo's modal inversion results

3.1 Calculation of Acoustic Impedance (Specimen A)

Fig. 4 shows the calculation results of the acoustic impedance Z_{tot}^* in the whole system and the acoustic impedance Z_{23}^* as viewed from the left end of the specimen calculated using Eqs. (2_15) and (2_16). In Fig. 4, Z_{tot}^* and Z_{23}^* are close to each other in the low-frequency range of 800 Hz or less. However, at higher frequencies, the difference in phase between them becomes more noticeable. At 1600 Hz, the phase of Z_{23}^* is delayed by 170° compared with that of Z_{tot}^*. In other words, the phase of Z_{23}^* is almost opposite in phase to Z_{tot}^*. This appears to be the influence of the reflection of the longitudinal wave at the interface between the left-most steel rod and the specimen in Fig. 3.

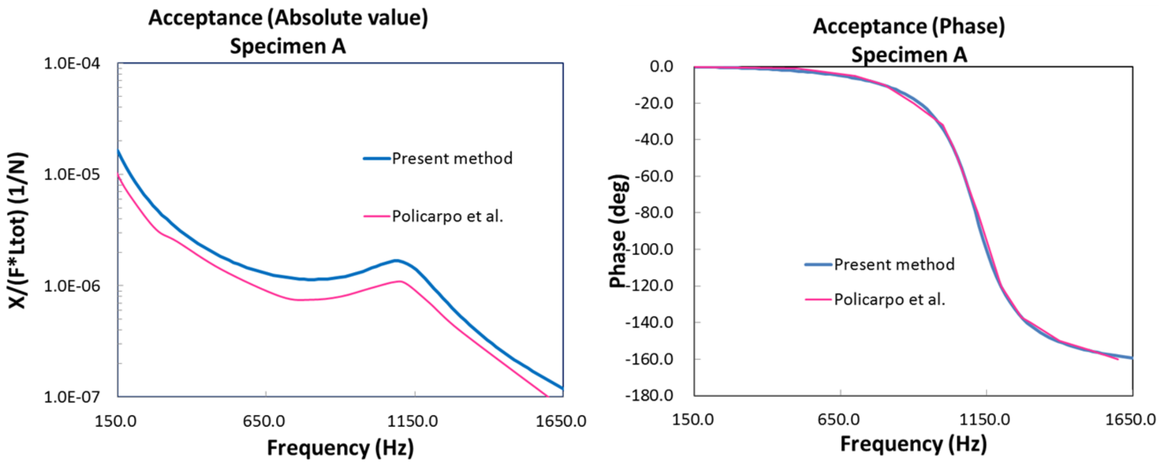


(a) Absolute value (b) Phase

Fig.4 Acoustic impedance of the whole system and the specimen (Specimen A)

3.2 Calculation of the Frequency Response Function (Specimen A)

We converted the acoustic impedance Z^*_{tot} of the whole system into the FRF $H_{specimen}$ (acceptance) by using Eqs. (2_19) to (2_21). In Fig. 5(a), the result obtained by dividing $H_{specimen}$ by the total length L_{tot} was compared with the results of Policarpo et al. (See (Note 3_1).) In Fig. 5 (b), the result obtained by subtracting 180° from the phase of $H_{specimen}$ was compared with the result of Policarpo et al. Our results are consistent with their results. Therefore, it can be concluded that the FRF using the acoustic impedance provides a reasonable result.



(a) Absolute value (b) Phase

Fig.5 FRF (acceptance) of the system (Specimen A)

(Note 3_1) Notation of force in the governing equation of Policarpo et al.

Policarpo et al. [4] utilized the following Eqs. (a) and (b) as the equation of motion and FRF H:

(Equation of motion)

$$E'_k A \frac{\partial^2 u_k}{\partial x^2} - \rho_k A \frac{\partial^2 u_k}{\partial t^2} = f(x,t) \quad \text{(a)}$$

(E'_k: storage modulus [Pa], u: displacement [m], A: cross-sectional area [m^2],
ρ_k: mass density [kg/m^3], x: location [m], t: time [s], f: force per unit length [N/m],
Subscript k : k^{th} element)

(Laplace-transformed equation of motion)

$$(\boldsymbol{M}s^2 + \boldsymbol{C}s + \boldsymbol{K})\boldsymbol{X}(s) = \boldsymbol{F}(s) \text{ i.e. } \boldsymbol{X}(s) = \boldsymbol{H}(s)\boldsymbol{F}(s), \quad \text{(b)}$$

where
$\boldsymbol{H}(s) = (\boldsymbol{M}s^2 + \boldsymbol{C}s + \boldsymbol{K})^{-1}$
(**M**,**C**,**K** : N × N mass, damping, and stiffness matrices, respectively , **F**: excitation force vector (the laplace transform of f(t))
Therefore, the following relation holds between the transfer function $H_{specimen}$ expressed by Eq. (2_21) and the FRF H of Policarpo et al.:

$$H = H_{specimen}/L_{tot} \quad , \tag{3\_1}$$

where
$L_{tot} = L_{SMP} + 2L_{steel}$

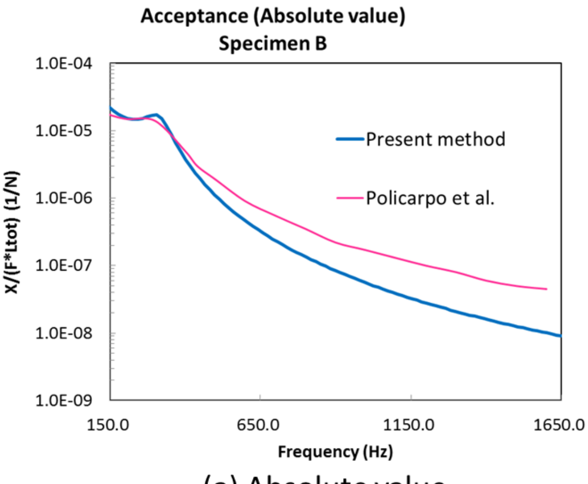
(Note 3_2) Phase inversion for $\mathbf{H_{specimen}}$
According to our numerical calculation, the reverse of the phase of the transfer function $H_{specimen}$ written in Eq.(2_21) coincides with the phase of the FRF H. Thus, the following relationship holds between the phases of both:

$$\arg(H) = \arg(H_{specimen}) - \pi \tag{3\_2}$$

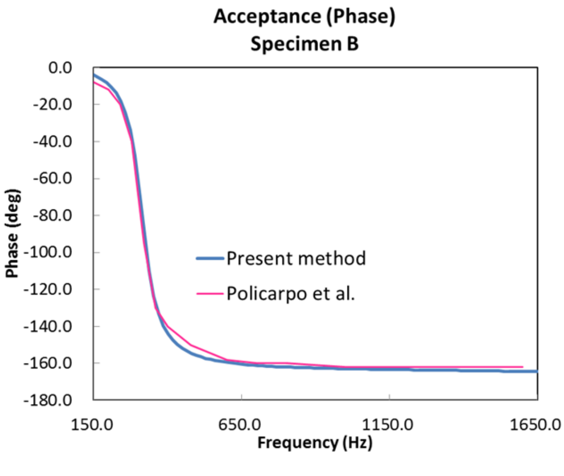
It appears that the force detected by the force transducer attached to the hammer is the reaction force of the impact force applied to the system.

3.3 Calculation of the Frequency Response Function (Specimen B)

In Fig.6, the FRF (acceptance) of specimen B is compared with the result of Policarpo et al. [4].In Fig. 6 (a), our FRF shows lower results in the frequency range above 300 Hz compared with the results of Policarpo et al.



(a) Absolute value



(b) Phase

Fig.6 FRF (acceptance) of the system (Specimen B)

Our modeling of wave attenuation is slightly different from their model in the following aspects. This is considered to be the cause of the difference.
(a) We calculate the acoustic impedance and FRF using the complex wave number expressed by Eqs.(2_10) and (2_11).

(b) The attenuation constant (factor) represents the degree of attenuation while the wave advances by unit length. When the loss factor is small (tan δ<< 1) in Eq.(2_11), the attenuation constant is proportional to tan δ as follows:

$$\alpha \approx \frac{\omega}{2}\left[\rho/E'^{(\omega)}\right]^{1/2} \ tan\delta \quad .$$

This approximation can be applied for specimen A. For specimen B with a large loss factor, the proportional relationship shown above no longer holds.

(c) The proportional relationship between the damping factor c and the loss factor is assumed in the modal-based analysis.

5. Conclusion

The acoustic impedance was defined for the propagation of the stress wave in a viscoelastic rod. To formulate the acoustic impedance, we used the formulas of the complex number derived in the previous study. In addition, we calculated the FRF (acceptance X / F) using the acoustic impedance. The FRF was verified via a comparison with the literature data of Policarpo et al.

It can be concluded that the FRF can be estimated from the viscoelastic data using the acoustic impedance for the longitudinal wave in the viscoelastic rod.

By using the acoustic impedance, not only the damping performance but also the sound absorption / sound insulation performance can be evaluated for a viscoelastic material such as cork.

Appendix A. Complex Acoustic Impedance of a Viscoelastic Rod

A1. Longitudinal Plane Wave in a Viscoelastic Rod

We deal with the longitudinal plane wave propagation in a viscoelastic rod. The equation of motion for steady state vibration, the stress–strain relation, and the displacement amplitude can be expressed as follows:

Equation of Motion

$$\frac{\partial^2 X}{\partial t^2} - \frac{1}{\rho}\frac{\partial \sigma}{\partial x} = 0 \qquad \text{(A1\_1)}$$

For a steady-state wave motion, the displacement can be expressed as follows:

$$X = Y(\omega, x)\cdot \exp\left(i\omega t\right) \qquad \text{(A1\_2)}$$

X: displacement [m], ρ :mass density [kg/m^3], $Y(\omega, x)$: Amplitude of the displacement [m],

$\sigma(t, x)$: Tensile stress [Pa], t: time [s]

[Response of the Tensile Stress to the Steady-state Vibration]

By applying the tensile stress to the Boltzmann's superposition, the response of the tensile stress can be expressed as follows:

$$\begin{aligned}
\sigma &= E_\infty \varepsilon + \sum_{k=1}^{N}\int_{-\infty}^{t}\frac{\partial \varepsilon(u,x)}{\partial u}E_k\cdot\exp\left(-\frac{t-u}{\tau_k}\right)du \\
&= \frac{\partial Y(\omega,x)}{\partial x}\left[E_\infty \exp(i\omega t) + \sum_{k=1}^{N}\int_{-\infty}^{t} i\omega\cdot\exp(i\omega u)E_k\cdot\exp\left(-\frac{t-u}{\tau_k}\right)du\right] \\
&= \frac{\partial Y(\omega,x)}{\partial x}\exp(i\omega t)\left[E_\infty + \sum_{k=1}^{N}\frac{E_k(\omega\tau_k)^2}{1+(\omega\tau_k)^2} + i\sum_{k=1}^{N}\frac{E_k\omega\tau_k}{1+(\omega\tau_k)^2}\right] \\
&= E^*(\omega)\frac{\partial Y(\omega,x)}{\partial x}\exp(i\omega t)
\end{aligned} \qquad \text{(A1_3)}$$

where

$\varepsilon = \partial X/\partial x$: tensile strain [-]

Complex Modulus

$$E^*(\omega)=E'(\omega)+iE''(\omega)\ ,$$

$$E'(\omega)=\sum_{k=1}^{N}\frac{E_k(\omega\tau_k)^2}{1+(\omega\tau_k)^2}\,,E''(\omega)=\sum_{k=1}^{N}\frac{E_k\omega\tau_k}{1+(\omega\tau_k)^2} \tag{A1\_4}$$

($E^*(\omega)$: complex modulus [Pa], $E'(\omega)$: storage modulus [Pa] , $E''(\omega)$: loss modulus [Pa])

[Response of the Displacement Amplitude]

Substituting Eqs. (A1_2)–(A1_3) into the equation of motion (Eq.(A1_1)), the amplitude equation of displacement is written in Eq.(A1_5). Further, the general solution of Eq.(A1_5) can be written in the form of Eq.(A1_6)

$$\omega^2Y(\omega,x)+\frac{E^*(\omega)}{\rho}\frac{\partial^2Y(\omega,x)}{\partial x^2}=0 \tag{A1\_5}$$

$$Y(\omega,x)=B_1\cdot\exp\left[-i\left(\rho/E^*(\omega)\right)^{\frac{1}{2}}\omega x\right]+D_1\cdot\exp\left[i\left(\rho/E^*(\omega)\right)^{\frac{1}{2}}\omega x\right],i=\sqrt{-1} \tag{A1\_6}$$

General solution, B_1, D_1:constants

A2. Acoustic Pressure and Particle Velocity in a Viscoelastic Rod

(Strain and Tensile Stress)

The displacement of the propagating wave in Eq.(A1_6) is expressed as follows:

Displacement

$$X(\omega,t,x)=B_1\cdot\exp\left[i\omega\left\{t-\left(\rho/E^*(\omega)\right)^{1/2}x\right\}\right]+D_1\cdot\exp\left[i\omega\left\{t+\left(\rho/E^*(\omega)\right)^{1/2}x\right\}\right] \tag{A2\_1}$$

For Eq. (A2_1),the linear tensile strain (engineering strain) and the tensile stress can be expressed as Eq.(A2_2) and Eq.(A2_3), respectively.

Linear Tensile Strain

$$\begin{aligned}&\varepsilon(\omega,t,x)\\&=\partial X/\partial x\\&=-iB_1\omega\left(\frac{\rho}{E^*(\omega)}\right)^{\frac{1}{2}}\cdot\exp\left[i\omega\left\{t-\left(\rho/E^*(\omega)\right)^{1/2}x\right\}\right]\ +iD_1\omega\left(\frac{\rho}{E^*(\omega)}\right)^{\frac{1}{2}}\cdot\exp\left[i\omega\left\{t+\left(\rho/E^*(\omega)\right)^{1/2}x\right\}\right]\end{aligned} \tag{A2\_2}$$

Tensile Stress

$$\sigma(\omega,t,x)=E^*(\omega)\varepsilon(\omega,t,x) \tag{A2\_3}$$

(Acoustic Pressure and Particle Velocity)

We will analyze the wave motion of the longitudinal wave in a viscoelastic rod in the same manner as in the acoustic tube. Accordingly, we identify the compressive stress ($-\sigma$) as the acoustic pressure p.

The “acoustic pressure” p defined below is not the real isotropic pressure. However, for a longitudinal wave in a viscoelastic rod, p can be identified as the ordinary acoustic pressure in air.

Acoustic Pressure

$$\begin{aligned}&p=-\sigma\\&=iB_1\cdot\omega\left[\rho E^*(\omega)\right]^{\frac{1}{2}}\cdot\exp\left[i\omega\left\{t-\left(\frac{\rho}{E^*(\omega)}\right)^{\frac{1}{2}}x\right\}\right]-iD_1\cdot\omega\left[\rho E^*(\omega)\right]^{\frac{1}{2}}\cdot\exp\left[i\omega\left\{t+\left(\frac{\rho}{E^*(\omega)}\right)^{\frac{1}{2}}x\right\}\right]\end{aligned} \tag{A2\_3}$$

In addition, the "particle velocity" u can be defined by differentiating Eq. (A2_1) with respect to time t. The "particle velocity" u can be identified as the ordinary particle velocity in air.

Particle Velocity

$$u = \partial X / \partial t$$
$$= i\omega B_1 \cdot \exp\left[i\omega\left\{t - \left(\rho / E^*(\omega)\right)^{1/2} x\right\}\right] + i\omega \cdot D_1 \cdot \exp\left[i\omega\left\{t + \left(\rho / E^*(\omega)\right)^{1/2} x\right\}\right] \qquad \text{(A2\_4)}$$

(Relation between Particle Velocity and Acoustic Pressure)

From Eqs. (A2_3) and (A2_4), the following relation holds between the particle velocity and the acoustic pressure.

(Formal Equation of the Acoustic Wave)

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad \text{(A2\_5)}$$

Eq. (A2_5) has the same form as the acoustic wave propagating in the acoustic tube. However, the attenuation effect of the wave contributes in the form of the attenuation constant α.

(Characteristic Impedance)

We deal with the wave propagating in a single direction (positive or negative x direction). In this case, the following relations hold between the particle velocity and the acoustic pressure.

Wave Propagating toward Positive X Direction (D=0)

$$p = \left[\rho E^*(\omega)\right]^{1/2} u \qquad \text{(A2\_6a)}$$

Wave Propagating toward Negative X Direction (B=0)

$$p = -\left[\rho E^*(\omega)\right]^{1/2} u \qquad \text{(A2\_6b)}$$

There are phase differences between p and u because of the complex modulus E^*.

Here, both the acoustic pressure p and the particle velocity u are expressed in complex numbers. The complex characteristic impedance can be defined as the ratio of the acoustic pressure and particle velocity in the above equation as follows:

$Z_C^*(\omega) = \left[\rho E^*(\omega)\right]^{1/2}$ **Complex Characteristic impedance** [Pa $\cdot\, s/m$=kg/(m^{2}s)] (A2_7)

Now, based on the expression of the characteristic impedance of air ($Z_0 = \rho_0 c_0$), the characteristic impedance is expressed in Eq.(A2_8a).

(ρ_0, c_0 : mass density $[kg/m^3]$ and speed of sound of air $[m/s]$, respectively)

$$Z_C^*(\omega) = \left[\rho E^*(\omega)\right]^{1/2} := \rho c^*(\omega) = \rho\left[c'(\omega) + ic''(\omega)\right] \qquad \text{(A2\_8a)}$$

Here, we introduced a new complex variable $c^*(\omega)$, which has the dimensions of velocity. This complex variable can be expressed using storage modulus and loss factor as follows:

$$c^*(\omega) = \left[E^*(\omega)/\rho\right]^{1/2} = \left[E'(\omega)/\rho\right]^{1/2}(1 + i\tan\delta)^{1/2} = \left[E'(\omega)/\rho\right]^{1/2}(\cos\delta)^{1/2}\left[\cos(\delta/2) + i\sin(\delta/2)\right] \qquad \text{(A2\_8b)}$$

Eqs. (A2_8) can be related to the real wave number and attenuation constant by using Eqs. (A2_9a, b) derived in the previous study [7].

Real Wave Number [1/m]

$$\beta(\omega) = \omega\left[\frac{\rho}{E'(\omega)}\right]^{\frac{1}{2}}\left[\cos\delta(\omega)\right]^{1/2}\cos\frac{\delta(\omega)}{2} = \omega\left[\frac{\rho}{2E'(\omega)}\right]^{\frac{1}{2}}\left[\frac{1}{(1+\tan^2\delta)^{1/2}} + \frac{1}{1+\tan^2\delta}\right]^{\frac{1}{2}} \qquad \text{(A2\_9a)}$$

Attenuation Constant (factor) [1/m]

$$\alpha(\omega) = \omega\left[\frac{\rho}{E'(\omega)}\right]^{\frac{1}{2}}\left[\cos\delta(\omega)\right]^{1/2}\sin\frac{\delta(\omega)}{2} = \omega\left[\frac{\rho}{2E'(\omega)}\right]^{\frac{1}{2}}\left[\frac{1}{(1+\tan^2\delta)^{1/2}} - \frac{1}{1+\tan^2\delta}\right]^{\frac{1}{2}} \qquad \text{(A2\_9b)}$$

From Eqs. (A2_8) and (A2_9), the real and imaginary parts of $c^*(\omega)$ can be expressed using the real wave number β and the attenuation constant α as follows:

$$c'(\omega) = \mathrm{Re}\left(Z_c^*/\rho\right) = \frac{\omega\beta}{\beta^2+\alpha^2}, c''(\omega) = \mathrm{Im}\left(Z_c^*/\rho\right) = \frac{\omega\alpha}{\beta^2+\alpha^2} \quad \text{(A2\_10)}$$

(Characteristic Impedance – Expression using the Complex Wave Number)

From Eqs. (A2_8) and (A2_10), the acoustic impedance $Z_c^*(\omega)$ can be expressed using the complex wave number as follows.

Relation between Characteristic Impedance and Complex Wave Number

$$Z_C^*(\omega) = \mathrm{Re}\left(Z_C^*\right) + i\,\mathrm{Im}\left(Z_C^*\right),$$
$$where \quad \text{(A2\_11)}$$
$$\mathrm{Re}\left(Z_C^*\right) = \frac{\rho\omega\beta}{\beta^2+\alpha^2}, \mathrm{Im}\left(Z_C^*\right) = \frac{\rho\omega\alpha}{\beta^2+\alpha^2}, i = \sqrt{-1}$$

Furthermore, from Eqs.(A2_11), the real wave number β and the attenuation constant α can be expressed using the real and imaginary parts of the complex characteristic impedance as follows.

Inverse Calculation of Eq. (A2_11): Formulas for Finding Complex Wave Number from Characteristic Impedance

$$\beta = \frac{1}{1+\left[\mathrm{Im}\left(Z_C^*\right)/\mathrm{Re}\left(Z_c^*\right)\right]^2}\cdot\frac{\rho\omega}{\mathrm{Re}\left(Z_C^*\right)},$$
$$\alpha = \frac{\mathrm{Im}\left(Z_C^*\right)/\mathrm{Re}\left(Z_C^*\right)}{1+\left[\mathrm{Im}\left(Z_C^*\right)/\mathrm{Re}\left(Z_C^*\right)\right]^2}\cdot\frac{\rho\omega}{\mathrm{Re}\left(Z_C^*\right)} \quad \text{(A2\_12a)}$$

Eqs. (A2_12a) can be used to determine the complex wave number in the material when the complex characteristic impedance of the specimen is experimentally obtained. The formulas (A2_12a) can also be rewritten as follows:

$$\beta = \rho\omega\cos\varphi/\left|Z_C^*\right|, \alpha = \rho\omega\sin\varphi/\left|Z_C^*\right|,$$
$$where \quad \text{(A2\_12b)}$$
$$\tan\varphi = \mathrm{Im}\left(Z_C^*\right)/\mathrm{Re}\left(Z_C^*\right)$$

Furthermore, with the inverse calculation (Eq. (A2_13)) of Eq. (A2_9), the storage modulus E' and the loss factor $\tan\delta$ of the material can be obtained from the complex wave number in the material as follows.

Formulas for Determining the Viscoelasticity $(E', tan\delta)$ of a Material from the Complex Wave Number $\beta^* = \beta - i\alpha$ in the Material

$$\tan\delta = \left[\left(\frac{\beta^2+\alpha^2}{\beta^2-\alpha^2}\right)^2 - 1\right]^{1/2},$$
$$E' = \frac{\rho\omega^2}{\beta^2+\alpha^2}\cdot\frac{1}{\left(1+\tan^2\delta\right)^{1/2}} = \rho\omega^2\frac{\beta^2-\alpha^2}{\left(\beta^2+\alpha^2\right)^2} \quad \text{(A2\_13)}$$

Continued use of Eqs. (A2_12) and (A2_13) makes it possible to identify the viscoelasticity $(E', tan\delta)$ of the material from the complex characteristic impedance $Z_c^*(\omega)$ of the material obtained experimentally.

(Note A_1) Derivation of Eqs.(A2_13)

From Eqs. (A2_9a, b), the sum and difference of the square of the real wave number and the attenuation constant can be expressed as follows. Eqs.(A2_13) is obtained by using the following:

$$\beta^2+\alpha^2=\frac{\rho\omega^2}{E'}\frac{1}{\left(1+\tan^2\delta\right)^{1/2}},\ \beta^2-\alpha^2=\frac{\rho\omega^2}{E'}\frac{1}{1+\tan^2\delta}$$

Appendix B. Acoustic Impedance of the Series Rods

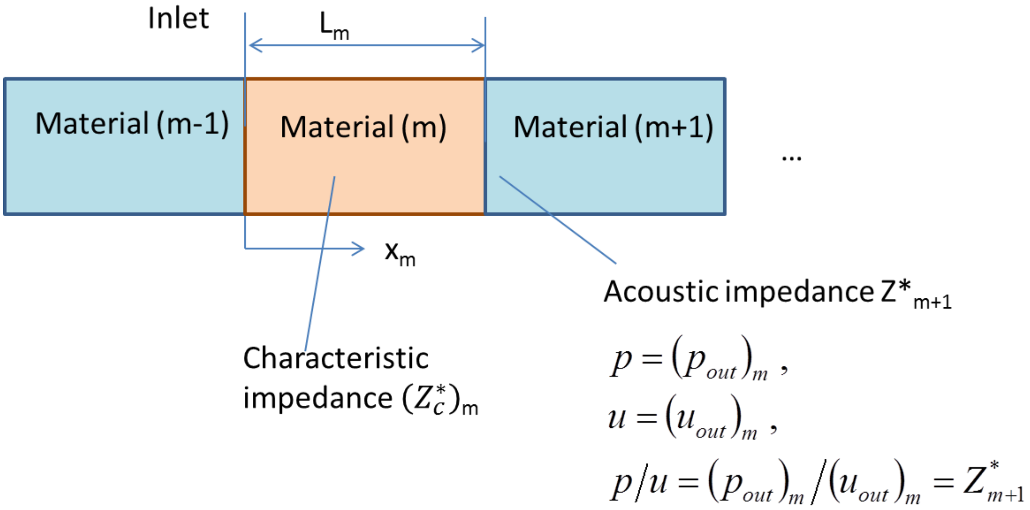


Fig B1.Series of rods

Let us consider the case where longitudinal waves propagate in series-connected viscoelastic rods as shown in Fig. B1. Based on the concept of Appendix A, the acoustic pressure and particle velocity in the material (m) are expressed as follows.

$$p=-i\rho_m\omega\left[B\exp\left\{i\left(\omega t-k_m^*x_m\right)\right\}+D\exp\left\{i\left(\omega t+k_m^*x_m\right)\right\}\right],$$
$$u=-ik_m^*\left[B\exp\left\{i\left(\omega t-k_m^*x_m\right)\right\}-D\exp\left\{i\left(\omega t+k_m^*x_m\right)\right\}\right], \qquad \text{(B1\_1)}$$

where

$k_m^*=\beta-i\alpha\ ,i=\sqrt{-1}$

x: the position along the material (m) from the edge of the material (m) (the boundary with the material (m – 1)) [m] ,

$k_m^*=\beta-i\alpha$: complex wave number in the material (m) [1/m]

In Fig. B1, let us set the following boundary condition to the right end ($x_m = L_m$) of the material (m).

Boundary Condition

$$p=\left(p_{out}\right)_m\exp(i\omega t),\ u=\left(u_{out}\right)_m\exp(i\omega t)\quad at\quad x_m=L_m \qquad \text{(B1\_2)}$$

At this time, following relations hold:

$$-i\rho\omega\left[B\exp\left(-ik_m^*L_m\right)+D\exp\left(ik_m^*L_m\right)\right]=\left(p_{out}\right)_m ,$$
$$-ik_m^*\left[B\exp\left(-ik_m^*L_m\right)-D_1\exp\left(ik_m^*L_m\right)\right]=\left(u_{out}\right)_m .$$

Therefore, the coefficients B and D are determined as follows:

$$B = \frac{i}{2}\frac{(u_{out})_m}{\rho\omega}\exp(ik_m^* L_m)\left[\frac{(p_{out})_m}{(u_{out})_m} + \frac{\rho\omega}{k_m^*}\right],\ D = \frac{i}{2}\frac{(u_{out})_m}{\rho\omega}\exp(ik_m^* L_m)\left[\frac{(p_{out})_m}{(u_{out})_m} - \frac{\rho\omega}{k_m^*}\right] \qquad \text{(B1\_3)}$$

In Eq.(B1_3), the following term is defined as the characteristic impedance of the material (m). (See Eq. (A2_11) in Appendix A)

Characteristic Impedance

$$(Z_c^*)_m := \frac{\rho\omega}{k_m^*} = \frac{\rho\omega}{\beta_m - i\alpha_m} = \frac{\rho\omega\beta_m}{\beta_m^2 + \alpha_m^2} + i\frac{\rho\omega\alpha_m}{\beta_m^2 + \alpha_m^2} \qquad \text{(B1\_4)}$$

Further, the ratio of the acoustic pressure and the particle velocity at the right end of the material (m) is defined as the acoustic impedance Z_{m+1}^*at the left end of the material (m+1).

Acoustic Impedance

$$Z_{m+1}^* := (p_{out})_m / (u_{out})_m \qquad \text{(B1\_5)}$$

When viewed from the left side of Fig. B1, Z_{m+1}^* is the acoustic impedance after the material (m + 1). Using equations (B1_4) and (B1_5), the coefficients B and D can be expressed as follows.

$$B = \frac{i}{2}\frac{(u_{out})_m}{\rho\omega}\exp(ik_m^* L_m)\left[Z_{m+1}^* + (Z_c^*)_m\right],\ D = \frac{i}{2}\frac{(u_{out})_m}{\rho\omega}\exp(-ik_m^* L_m)\left[Z_{m+1}^* - (Z_c^*)_m\right] \qquad \text{(B1\_6)}$$

(Acoustic Impedance at the Left End of the Material (m))

The acoustic impedance at the left end $(x_m = 0)$ of the material (m) can be expressed as follows:

$$\begin{aligned} Z_m^* &:= (p)_{xm=0} / (u)_{xm=0} \\ &= \frac{\rho\omega}{k_m^*}\cdot\frac{B+D}{B-D} \\ &= (Z_C^*)_m \frac{\exp(ik_m^* L_m)\left[Z_{m+1}^* + (Z_c^*)_m\right] + \exp(-ik_m^* L_m)\left[Z_{m+1}^* - (Z_c^*)_m\right]}{\exp(ik_m^* L_m)\left[Z_{m+1}^* + (Z_c^*)_m\right] - \exp(-ik_m^* L_m)\left[Z_{m+1}^* - (Z_c^*)_m\right]} \\ &= (Z_C^*)_m \frac{(Z_c^*)_m \sinh(ik_m^* L_m) + Z_{m+1}^* \cosh(ik_m^* L_m)}{(Z_c^*)_m \cosh(ik_m^* L_m) + Z_{m+1}^* \sinh(ik_m^* L_m)} \end{aligned} \qquad \text{(B1\_7)}$$

Eq. (B 1_7) is in the form of a recurrence formula related to the section number m.

(Distribution of the Acoustic Pressure and Particle Velocity in the Material (m))

To obtain a transfer function in the form of (velocity) / (force) between the ends of the material (m), it is necessary to clarify the distribution of acoustic pressure and particle velocity in the material (m). Substituting Eq. (B1_ 6) into Eq. (B1_1), the acoustic pressure and particle velocity distribution within the material (m) can be expressed as follows:

$$\begin{aligned} p &= -i\rho\omega\exp(i\omega t)\frac{i(u_{out})_m}{2\rho\omega}\left[\left(Z_{m+1}^* + (Z_c^*)_m\right)\exp\{ik_m^*(L_m - x_m)\} + \left(Z_{m+1}^* - (Z_c^*)_m\right)\exp\{-ik_m^*(L_m - x_m)\}\right], \\ &= (u_{out})_m \exp(i\omega t)\left[Z_{m+1}^* \cosh\{ik_m^*(L_m - x_m)\} + (Z_c^*)_m \sinh\{ik_m^*(L_m - x_m)\}\right] \quad , \\ u &= -ik_m^* \exp(i\omega t)\frac{i(u_{out})_m}{2\rho\omega}\left[\left(Z_{m+1}^* + (Z_c^*)_m\right)\exp\{ik_m^*(L_m - x_m)\} - \left(Z_{m+1}^* - (Z_c^*)_m\right)\exp\{-ik_m^*(L_m - x_m)\}\right] \\ &= (u_{out})_m \exp(i\omega t)\left[\cosh\{ik_m^*(L_m - x_m)\} + \frac{Z_{m+1}^*}{(Z_c^*)_m}\sinh\{ik_m^*(L_m - x_m)\}\right] \end{aligned}$$

(B1_8)

(Acoustic Pressure Ratio and Particle Velocity Ratio at Both Ends of the Material (m))
Using Eq. (B1_ 8), the acoustic pressure ratio and the particle velocity ratio at both ends of the material (m) are given as follows:

$$\frac{(p)_{xm=0}}{(p)_{xm=Lm}}=\frac{Z^*_{m+1}\cosh\left(ik^*_mL_m\right)+\left(Z^*_c\right)_m\sinh\left(ik^*_mL_m\right)}{Z^*_{m+1}}=\cosh\left(ik^*_mL_m\right)+\frac{\left(Z^*_c\right)_m}{Z^*_{m+1}}\sinh\left(ik^*_mL_m\right) \qquad \text{(B1\_9)}$$

$$\frac{(u)_{xm=0}}{(u)_{xm=Lm}}=\frac{\left(Z^*_{m+1}+\left(Z^*_c\right)_m\right)\exp\left(ik^*_mL_m\right)-\left(Z^*_{m+1}-\left(Z^*_c\right)_m\right)\exp\left(-ik^*_mL_m\right)}{\left(Z^*_{m+1}+\left(Z^*_c\right)_m\right)-\left(Z^*_{m+1}-\left(Z^*_c\right)_m\right)}$$
$$=\cosh\left(ik^*_mL_m\right)+\frac{Z^*_{m+1}}{\left(Z^*_c\right)_m}\sinh\left(ik^*_mL_m\right) \qquad \text{(B1\_10)}$$

Appendix C. Acoustic Impedance of the Steel Rod in contact with the Backed Air
C1. Acoustic Impedance of the Backed Air
In the case of the apparatus in Fig. 1, the right end of the apparatus is in contact with open air. In this case, the acoustic impedance viewed from the right end of the device is equal to the characteristic impedance of air $Z_0(=\rho_0c_0)$.
where

$\rho_0=p_aR_aT_a$: Mass density of air $\left[\mathrm{kg/m^3}\right]$, $c_0=\sqrt{\gamma R_aT_a}$: Speed of sound of air $[\mathrm{m/s}]$,
$R_a=289.03$: Gas constant of air $\left[J/(kgK)\right]$, $\gamma=1.4$: Specific heat ratio $[-]$,
$p_a=1.0133\times10^5$: Atmospheric pressure $[Pa]$, T_a : Temperature $[K]$

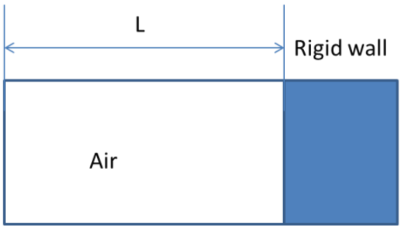


Fig.C1. Air gap backed by a rigid wall

This can be explained from the case shown in Fig. C1. In Fig. C1, a rigid wall is placed behind an air layer of length L. Let us consider the case where L becomes infinitely large in Fig. C1. First, as the particle velocity at the surface of the rigid wall is 0, the acoustic impedance Z_{bRW} as viewed from the left end of the air layer of Fig. C1 is given by the following expression:

$$Z_{bRW}=-iZ_0\cot\left(\omega L/c_0\right) \qquad \text{(C1\_1)}$$

If air is regarded as a medium having no viscosity, even if L → ∞, the right-hand side of Eq. (C1_1) does not converge. Thus, let us add the attenuation of the acoustic wave as follows:

$$k^*=\beta_A-i\alpha_A\ ,$$
$$where \qquad \text{(C1\_2)}$$
$$\beta_A=\omega/c_0\ ,\ \alpha_A>0$$

α_A :Attenuation constant [1/m], k^*: complex wave number [1/m]

In this case, the acoustic impedance Z_{bRW} is rewritten as follows:

$$Z_{bRW} = -iZ_0 \cot[(\beta_A - i\alpha_A)L] = Z_0 \frac{\exp(i\beta_A L)\exp(\alpha_A L) + \exp(-i\beta_A L)\exp(-\alpha_A L)}{\exp(i\beta_A L)\exp(\alpha_A L) - \exp(-i\beta_A L)\exp(-\alpha_A L)} \to Z_0 \quad \text{(C1\_3)}$$

$$as \quad L \to \infty$$

Therefore, for infinitely large L, the acoustic impedance Z_{bRW} simply becomes the characteristic impedance Z_0 of air.

C2. Acoustic Impedance of Steel Rod in Contact with the Backed Air

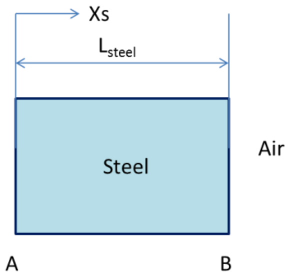


Fig.C2 Steel rod in contact with the backed air

Let us consider the acoustic impedance viewed from the left end A of the steel rod whose right end is in contact with the atmosphere (infinite air). Replace the characteristic impedance of the material with $(Z_c^*)_m = Z_{steel}$ in Eq. (B1_7) and replace the wavenumber in the material with $k_m^* = \omega/c_{steel}$. ($c_{\ steel}$: sound velocity of steel). At this time, the acoustic impedance viewed from the left end (point A in the figure) of the steel rod can be expressed as follows:

$$Z_{Sb}^* = Z_{Steel} \cdot \frac{Z_0 \cos(\omega L_{Steel}/c_{Steel}) + iZ_{Steel}\sin(\omega L_{Steel}/c_{Steel})}{iZ_0 \sin(\omega L_{Steel}/c_{Steel}) + Z_{Steel}\cos(\omega L_{Steel}/c_{Steel})}$$

$$= Z_{Steel} \cdot \frac{\dfrac{Z_0}{Z_{Steel}} \cos(\omega L_{Steel}/c_{Steel}) + i\sin(\omega L_{Steel}/c_{Steel})}{i\dfrac{Z_0}{Z_{Steel}} \sin(\omega L_{Steel}/c_{Steel}) + \cos(\omega L_{Steel}/c_{Steel})} \cong iZ_{Steel} \tan\left(\frac{\omega L_{Steel}}{c_{Steel}}\right), \quad \text{(C2\_1)}$$

where

$$c_{Steel} = \sqrt{\frac{E_{Steel}(1-\nu_{Steel})}{\rho_{Steel}(1+\nu_{Steel})(1-2\nu_{Steel})}}$$

(E_{Steel}: Young's modulus of steel [Pa], ν_{Steel}: Poisson's ratio of steel [-], ρ_{Steel} : mass density of steel [kg/m^3])

The approximate equality in Eq. (C2 - 1) is because $Z_{steel} \gg Z_0$.

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